Network Formation with Multigraphs and Strategic Complementarities*

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Abstract

This paper examines the formation of one network **G** when connections in a second network **H** are inherited under two scenarios: (i) **H** is asymmetric allowing for a wide range of networks called nested split graphs, and (ii) **H** is symmetric in Bonacich centrality. The bulk of our paper assumes that both **G** and **H** are interdependent because the respective actions in each are (weak) strategic complements. This complementarity creates a "silver spoon" effect whereby those who inherit high Bonacich centrality in **H** will continue to have high Bonacich centrality in **G**. There is however a "silver lining": depending on the costs of link formation, the formed network **G** may allow for an improvement in centrality. As an application, we introduce an overlapping generations models to analyze intergenerational transmission of inequality through networks. Finally, we explore the implications of actions being strategic substitutes across networks. This can lead to a "leisure class" à la Veblen where well connected agents in **H** establish no links in **G**, and those with no connections in **H** form all the links in **G**. Our analysis provides insight into preferential attachment, how asymmetries in one network may be magnified or diminished in another, and why players with links in one network may form no links in another network.

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1 Introduction

The typical economic agent is concurrently engaged in many different relationships. Individuals can simultaneously belong to both social and business networks, firms can be involved in multiple collaborative relationships along different dimensions of R&D, and countries can be signatories to multiple economic and/or military alliances. This raises an important question: When agents straddle multiple networks, then to what extent are an agent's connection in one network shaped by the connections in others? This question has largely been left unaddressed in the literature which has attempted to explain the architecture of equilibrium networks in terms of the benefits and costs of links established *within* a network without any reference to the role of other networks. In this paper we allow for the co-existence of multiple network relationships and delineate the role of one network in shaping the equilibrium architecture of another.

We develop a formal "multigraph" model to explain these types of situations. We assume that agents are participants in two networks. To fix ideas let us call one of these the network of *social connections* (denoted by $\mathbf{H}$), and the other one the network of economic activities, or the *business network* (denoted by $\mathbf{G}$). Each agent's utility is a function of what happens in both networks. Following Ballester et al (2006), the utility function of each agent exhibits (local) complementarities in actions with others to whom the player is linked within a network. Moreover, to model interaction between the two networks, in much of the paper we assume that the actions of a player across networks are also complementary. This captures the notion that the amount of time and energy devoted towards social and business networks are strategic complements. For instance, a person who is well known in her social network may be able to secure credit on better terms, or have access to better business opportunities.[1] This could be due to favors given, or the fact that increased engagement in one network might facilitate working together in another network, making the efforts exerted in these cases complementary across networks.

In our analysis, we assume that the network of social connections, $\mathbf{H}$, is fixed. An intuitive way to think about this is that agents inherit their social connections. We believe this is a realistic assumption that makes multigraph network formation tractable from an analytical perspective. Alternatively, this can be also viewed as a way to understand the role of feedback effects when the formation of multiple networks is not simultaneous but asynchronous. In other words, it allows us to capture the role of path dependence in network formation making it especially useful for answering questions about inequality in networks. Hence, in our model, link formation occurs only in the network $\mathbf{G}$, but the results show how the social network $\mathbf{H}$ affects link formation in $\mathbf{G}$. We view the network formation game in $\mathbf{G}$ as a two-stage process. In the first stage players form their network $\mathbf{G}$, and in the second stage agents utilize the two networks by choosing actions in both networks that determines payoffs. The second stage game is a standard Nash

[1] Biggs, Raturi and Shrivastav (2002) find that in Kenya, a borrower's membership in ethnic networks does not play a role in access to commercial credit, but is important for supplier credit, especially among Asian groups. They argue that this is due to the fact that ethnic groups have better information and contract enforcement mechanisms. Fafchamps (2000) documents a similar outcome using data from manufacturing firms in Kenya and Zimbabwe.

game in actions, conditional on the networks from the first stage. The first stage network formation game is modeled as a sequential game starting from the empty network. When it is an agent's turn to move, she can delete (any number of) links as well as propose one (and only one) link in **G** to an unlinked agent from whom she gets the highest incremental utility; the link is formed if the potential partner accepts the proposal. Agents in the game move repeatedly till no agents wishes to make any more moves. The network formation game is shown to generate an "improving path" of networks which converges to a pairwise-stable equilibrium. The *pairwise-stable equilibrium*, proposed by Goyal and Joshi (2006), corresponds to a network **G** in which no agent has an incentive to delete links and no pair of unlinked agents can mutually profit from a link.

Strategic complementarities in actions, both within and across networks, create feedback loops that need to be bounded to ensure the existence of a Nash equilibrium in the second stage. This requires that the degree of complementarity in actions within networks as well as across the two networks be sufficiently small, making the linkage between the two networks relatively weak. Our result however is quite striking, in which the architecture of **H** has an important bearing on the equilibrium structure of **G**. To understand this we consider two extreme architectures for the fixed network **H**: *Nested split graphs* (or NSG), and a subclass of *regular* graphs. In a NSG the neighborhoods, or the set of direct connections of agents, are nested allowing for a large class of asymmetric networks. Suppose we partition a network into groups using the degrees, or number of direct connections, of the players. Then one way to think about a NSG is that the neighborhood of every agent in a particular partition subsumes the neighborhood of all the agents in partitions below (i.e. sets of agents with fewer degrees). The star network, for instance, is a NSG. Examples are provided in figure 1 along with their respective degree partitions. The other kind of network we consider is a subclass of *regular graphs* in which all agents have the same Katz-Bonacich centrality (henceforth *KB-regular*) giving rise to a highly symmetric architecture. Recall that regular graphs are networks where every node has the same degree. Intuitively, our subclass of KB-regular graphs requires that every node has the same level of importance in the network.

Our main result (Theorem 1) is the following. If the inherited network **H** is a NSG, then the endogenously formed network **G** is either empty, complete, or a NSG with at most one (non-singleton) component. Moreover, agents who have the most social connections in **H** also have the most connections in **G**, and those with the fewest connections in **H** have the fewest connections in **G**. We call this type of preferential attachment the *silver spoon effect*, i.e. those who are privileged in terms of their social connections benefit more in the business network as well. On the other hand, when **H** is a KB-regular graph, then **G** is either empty or complete. By allowing **G** and **H** to be related through (weak) strategic complementarities in actions, we provide an explanation for how inequality among agents maybe entrenched, with inequality in one network of connections (like NSG) being mirrored across another network of connections. As an application of our model, we introduce a simple overlapping generations model where each generation is replaced by exactly another generation of the same size. Each generation inherits the professional network of its parents (as its social network) and forms its own professional network which then becomes the social network of the succeeding generation. Assuming action complementarity across generations, we find that starting with a NSG, inequality in connections can persist. We identify costs of link formation under which **G** will either be an empty or a complete network. From a development perspective, our analysis sheds light on how class structures can be preserved in societies. Those born into privileged classes with high social connections have access to better business opportunities allowing them to do better. This may also explain why those with fewer connections may find it harder to have access to a larger set of economic opportunities and find it difficult to escape poverty traps.

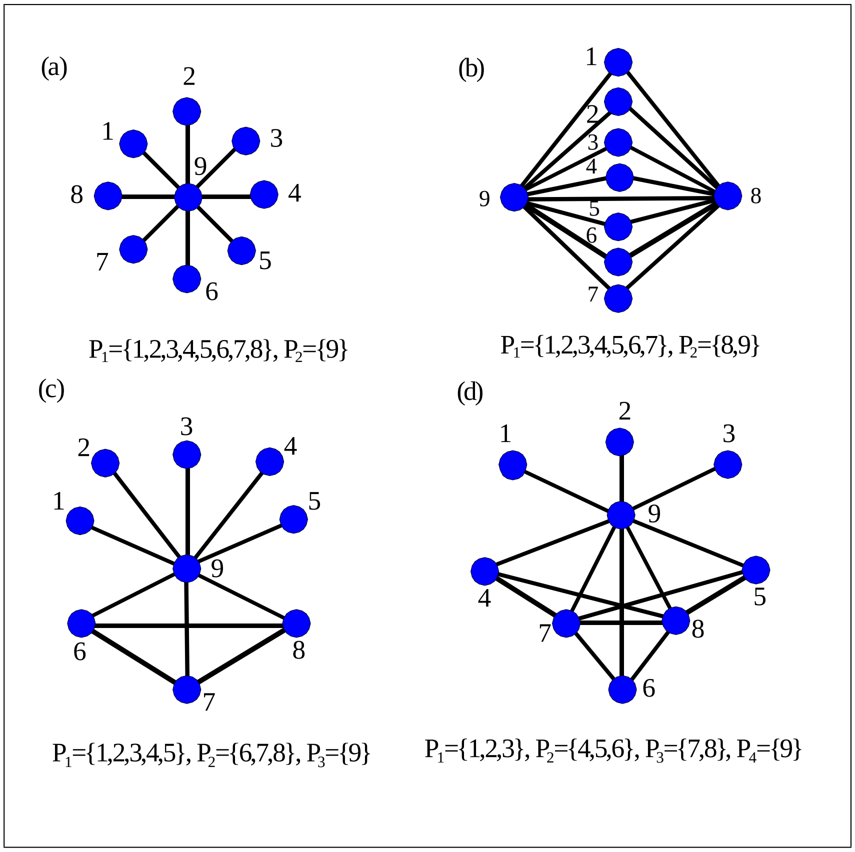

Figure 1: Nested Split Graphs and their Degree Partitions

The persistence of inequality can be explained as follows. The reduced form incremental utility of an agent from forging a link is a function of the partner's Katz-Bonacich (KB) centrality[2] in network **H** and the neighborhood in network **G** (Lemma 1). In the sequential network formation game, this creates an incentive for each agent to link to a partner with the highest KB centrality in **H**. Starting from the empty network, this preferential attachment generates an improving path along which the most central agents in **H** have neighborhoods in **G** that nest the neighborhoods of less central players in **H**. Strategic complementarity obviates the incentive to delete any existing links and therefore the improving path converges to a limiting pairwise-stable equilibrium network that is also a NSG.

[2] The Katz-Bonacich centrality of an agent counts the number of weighted walks in the network **H** originating from the agent, with the weights on the walks falling exponentially with their length. It is a measure of the prominence of the agent in network **H**.

Our analysis also offers a perspective on the role played by costs of link formation in magnifying or diminishing inherited inequalities in the process of network formation. If **H** is a NSG, then **G** assumes a *coarser* NSG architecture with a higher or lower level of inequality in the number of direct connections depending on the cost of link formation. We show that when linking costs exceed a certain threshold, then the degree distribution of **G** "shifts down" in the sense of first order dominance relative to **H** (Proposition 5). To the extent that players at the lower extreme of the degree distribution see a strict decrease in the number of their direct connections, the inequality inherent in **H** is amplified. The opposite is true when linking costs fall below a certain threshold. Additionally, our analysis can predict the *identity* of agents occupying various nodes in an asymmetric equilibrium network. Based on the positions of agents in **H**, we can characterize both the equilibrium network structure of **G** *and* the identity of agents occupying the vertices of **G**.

Finally, we also explore the implications of actions across the networks being strategic substitututes. Given that there is now a tradeoff in actions across the two networks, strategic substitutability affects who the agents will form links with and therefore the equilibrium networks. We find the the earlier relationship between centrality and neighborhood sizes is reversed. Now i benefits more from linking to the agent with the larger neighborhood in **G** and (given the negative spillovers) prefers this agent to have lower centrality in **H**. In terms of equilibrium networks, a complete charecterization is difficult. Instead, we identify conditions that will lead to the empty and complete networks. We also find the existence of a phenomenon akin to Veblen's "theory of the leisure class." Suppose that in **H** there exists two sets of agents: those who are part of a completely connected component and others who are isolated. We find a sufficient condition under which this network gets "inverted", i.e. in **G**, the isolated agents form a complete component while the other agents form no links and are now isolated!

Our paper intersects with several literatures. A substantial body of empirical evidence supports the dependence of one network of relationships on other networks. For instance, social networks can have a strong influence on business networks. Belonging to specific clubs, being a member of certain social groups or attending certain events in the real or virtual world can play a key role in business affairs. This link between social and business relationships has been well documented for the *guanaxi* in China (Kali, 1999), the *chaebols* in Korea, and several communities like the Marwaris and Parsis in India (Damodaran, 2008). This has even been studied in the context of networks by examining marriage alliances. Padgett and Ansell (1993) examine marriages between 16 elite families in Florence during the early 1400s to explain the emergence of the Medici as an economic force. They document that the Medici's rise in economic and political circles can be traced to their increasing importance in the marriage network and the advantages it conferred with respect to communicating information, brokering business deals, and reaching political decisions. Similarly, Munshi (2011) demonstrates that a tight community network can help a poor community leap-frog ahead as long as some members of the disadvantaged community have access to other networks of opportunities.

There is a significant body of work outside economics that deals with multiple networks, also referred to as *multiplex* networks.[3] Much of this literature is rooted in computer science, is simulation driven, and does not explicitly model strategic issues.[4] The paper in the economics literature that is closest to our study is recent work of Chen et al. (2016). Players in their model engage in two different activities which can be complements or substitutes and therefore can be seen as two different types of relationships. However, the authors take the network as given, and do not address the network formation problem. Apart from the focus on network formation, we differ from this literature in two main ways: (i) by assuming that the networks are not exogenously given, and (ii) by incorporating strategic behavior.

The paper is organized as follows. Section 2 contains the preliminaries needed for the analysis and section 3 presents the strategic complementarities model. Section 4 discusses the issue of unequal distribution of links. In section 5 we introduce an application in the form of an overlapping generations model. Section 6 explores strategic substituability in actions, and section 7 concludes. All proofs are collected in an appendix.

2 Preliminaries

Networks/Graphs

A *network* (or *graph*) is a tuple $(\mathcal{N}, \mathbf{H})$, where $\mathcal{N} = \{1, 2, ..., N\}$ denotes the set of players, and $\mathbf{H}$ records the undirected links that exist between the players. When $\mathcal{N}$ is unambiguous, we refer to $\mathbf{H}$ as the network, and represent it in two alternative ways. The first is by letting $\mathbf{H}$ denote the collection of all pairwise links that exist in the network such that if $ij \in \mathbf{H}$, then $ji \in \mathbf{H}$. The second is by letting $\mathbf{H} = [h_{ij}]$ denote the (symmetric) adjacency matrix of the network such that $h_{ij} = h_{ji} = 1$ if i and j are linked and $h_{ij} = h_{ji} = 0$ otherwise. Additionally, $h_{ii} = 0 \; \forall i \in \mathcal{N}$. The set $\mathbf{N}_i(\mathbf{H}) = \{j \in \mathcal{N} \backslash \{i\} : ij \in \mathbf{H}\}$ denotes the *neighborhood*, and $\overline{\mathbf{N}}_i(\mathbf{H}) = \mathbf{N}_i(\mathbf{H}) \cup \{i\}$ the *augmented* neighborhood, of player i in $\mathbf{H}$. Player i's *degree* is $d_i(\mathbf{H}) = |\mathbf{N}_i(\mathbf{H})|$ and $\mathcal{D}(\mathbf{H}) = \{d_1(\mathbf{H}), d_2(\mathbf{H}), ..., d_N(\mathbf{H})\}$ the degree distribution in $\mathbf{H}$. In a *regular* network, $d_i(\mathbf{H})$ is the same for all players, which is also the degree of the network. Examples include the *complete* network, $\mathbf{H}^c$ (degree $N - 1$) and the *empty* network, $\mathbf{H}^e$ (degree 0). Let the distinct positive degrees in $\mathbf{H}$ be $d_{(1)} < d_{(2)} < \cdots < d_{(m)}$ with $d_{(0)} = 0$, even though there may not exist an isolated player. The *degree partition* of $\mathbf{H}$ is denoted by $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}), P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$, where $P_k(\mathbf{H}) = \{i \in \mathcal{N} : |\mathbf{N}_i(\mathbf{H})| = d_{(k)}\}$, $k \in \{0, 1, ..., m\}$.

A *walk* in $\mathbf{H}$ connecting i and j is a set of nodes $\{i_1, \ldots, i_n\}$ such that $ii_1, i_1i_2, ..., i_{n-1}i_n, i_nj \in \mathbf{H}$. A *chain* is a walk in which all nodes are distinct.[5] A network is *connected* if there exists a chain between any pair $i, j \in \mathcal{N}$; otherwise the network is *unconnected*. A sub-network, $C(\mathbf{H}) \equiv (\mathcal{N}', \mathbf{H}')$, $\mathcal{N}' \subset \mathcal{N}$, $\mathbf{H}' \subset \mathbf{H}$, is a

[3] In contrast to the term multiplex networks which primarily focus on networks of different relationships, we have chosen to use the term "multigraph" to capture the notion of network formation.

[4] Often the focus is on issues like identifying communities in exogenously given multiplex networks (Mondragon et al. (2017)) or examining co-evolution of multiplex networks based on a pre-specified feedback mechanism (Wu et al. 2017). Although there are a few papers examining the evolution of cooperation in multiplex networks (Gómez-Gardeñes et al. (2012)), they usually assume that the networks are exogenously given. Similarly, there are papers relating to multiplex interbank networks, which are again devoid of strategic elements (Bargigle et al. (2015)).

[5] A walk with distinct nodes is also called a *path*. We follow Jackson and Watts (2002) in using the term "path" when referring to a sequence of networks.

component of the network $(\mathcal{N}, \mathbf{H})$ if it is connected, and if $ij \in \mathbf{H}$ for $i \in \mathcal{N}'$, $j \in \mathcal{N}$, implies $j \in \mathcal{N}'$ and $ij \in \mathbf{H}'$. Let $\mathbf{H} - ij$ (respectively $\mathbf{H} + ij$) denote the network obtained from $\mathbf{H}$ by deleting (respectively adding) the link ij.

Nested split graphs (NSG) play an important role in our paper and are defined along the lines of König et al. (2014). Let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to x. A network $\mathbf{H}$ with degree partition $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}), P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$ is a NSG if for each player $i \in P_k(\mathbf{H})$, $k \in \{1, 2, .., m\}$:

$$\mathbf{N}_i(\mathbf{H}) = \begin{cases} \cup_{l=1}^{k} P_{m+1-l}(\mathbf{H}), & k = 1, 2, ..., \lfloor m/2 \rfloor \\ \cup_{l=1}^{k} P_{m+1-l}(\mathbf{H}) \backslash \{i\}, & k = \lfloor m/2 \rfloor + 1, ..., m \end{cases}$$

A *star*, $\mathbf{H}^s$, is a special case of a NSG where $|P_1(\mathbf{H}^s)| = N - 1$ and $|P_2(\mathbf{H}^s)| = 1$. Another special case is a $k-$*dominant group* network, $\mathbf{H}^{d_k}$, comprising of $k \in \{2, 3, ..., N-1\}$ players who form a complete component and $N - k$ isolated players, i.e. $\left|P_0(\mathbf{H}^{d_k})\right| = N - k$ and $\left|P_1(\mathbf{H}^{d_k})\right| = k$.

Katz-Bonacich Centrality

Given the identity matrix $\mathbf{I}$ and a sufficiently small $\xi > 0$, consider the matrix:

$$\mathbf{M}(\mathbf{H},\xi) \equiv [\mathbf{I}-\xi\mathbf{H}]^{-1} = \sum_{s=0}^{\infty} \xi^s \mathbf{H}^s \tag{1}$$

where $\mathbf{H}^0 = \mathbf{I}$. If $\xi\mu_{\max}(\mathbf{H}) < 1$, where $\mu_{\max}(\mathbf{H}) > 0$ is the spectral radius of $\mathbf{H}$, then $[\mathbf{I} - \xi\mathbf{H}]^{-1}$ is well-defined and non-negative. For any $N \times 1$ vector $\mathbf{a}$, the vector of $\mathbf{a}$-weighted *Katz-Bonacich* (*KB*) *centralities* (Bonacich, 1987) of the players is given by:

$$\mathbf{b}\left(\mathbf{H},\xi, \mathbf{a}\right) = \mathbf{M}(\mathbf{H},\xi)\mathbf{a} = \sum_{s=0}^{\infty} \xi^s \mathbf{H}^s \mathbf{a} \tag{2}$$

When $\mathbf{a} = \mathbf{1}$, we get the unweighted KB centralities $\mathbf{b}\left(\mathbf{H},\xi, \mathbf{1}\right)$ for the players. The i^{th}-component, $b_i(\mathbf{H},\xi, \mathbf{1})$, measures the total number of ξ-weighted walks originating from player i in the network $\mathbf{H}$. Note that for a NSG, *the partition of players according to their KB centralities coincides with their degree partition* (König et al., 2014, Proposition 1).

KB-regular Graphs

To contrast the impact on network formation when the inherited network $\mathbf{H}$ is a highly asymmetric NSG, we consider the case where $\mathbf{H}$ is highly symmetric. In particular, we consider a subclass of regular graphs, called *KB-regular* graphs, in which all players have the same KB centrality. Regular connected graphs are also KB-regular, but unconnected regular graphs with asymmetrically-sized components are not KB-regular.

Timing of moves

In our model players interact with each other through two different sets of connections. The network $\mathbf{G}$ keeps track of players' links along one sphere, say economic, while network $\mathbf{H}$ accounts for links along another dimension, say the social. Thus, in our model, interactions between the players are captured by two separate adjacency matrices. We study a two-stage game. In the first stage we assume that: (i) network $\mathbf{H}$ is (historically) given and does not change, and (ii) players form the network $\mathbf{G}$. In the second stage players play a Nash game contingent on $\mathbf{G}$ and $\mathbf{H}$.

Payoffs

Let a compact set $A_i \subset \mathbb{R}^2_+$ denote the action set of player i for the second stage Nash game where, for $a_i = (x_i, y_i) \in A_i$, we let x_i denote player i's action in network $\mathbf{G}$ and y_i in network $\mathbf{H}$. Let $A = \times_{j=1}^n A_j$ denote the set of action profiles and $u_i : A \times \mathbf{G} \times \mathbf{H} \to \mathbb{R}$ the utility function of player i. Following Ballester et al. (2006), we specify a linear-quadratic utility function:

$$u_i(\mathbf{a}_i, \mathbf{a}_{-i}; \mathbf{G}, \mathbf{H}) = \left[x_i - \frac{1}{2} x_i^2 + \lambda x_i \sum_{j \in \mathcal{N}} g_{ij} x_j \right] + \left[y_i - \frac{1}{2} y_i^2 + \psi y_i \sum_{k \in \mathcal{N}} h_{ik} y_k \right] + \gamma x_i y_i \tag{3}$$

The first two terms within square brackets capture the impact of own actions in networks $\mathbf{G}$ and $\mathbf{H}$ respectively. The negative quadratic within each bracket represents the cost of action in that network. We assume that the parameters satisfy $\lambda > 0$ and $\psi > 0$ so that the actions within each network are strategic complements with those of other players. In order to remain agnostic *a priori* about link formation in $\mathbf{G}$, we assume that λ is sufficiently small. Ensuring that strategic complementarities within $\mathbf{G}$ are sufficiently weak permits us to draw out the role played by the topology of $\mathbf{H}$ on the network formation process in $\mathbf{G}$. The last term is crucial in that it creates interdependencies between networks $\mathbf{G}$ and $\mathbf{H}$ by creating strategic complementarity between actions of player i in the two networks when $\gamma > 0$ and strategic substitutability when $\gamma < 0$.

Second Stage Nash Equilibrium

Given networks $\mathbf{G}$ and $\mathbf{H}$, an action profile $\mathbf{a}^*(\mathbf{G}, \mathbf{H})$ is a Nash equilibrium of the second stage game if :

$$u_i(a_i^*(\mathbf{G}, \mathbf{H}), \mathbf{a}_{-i}^*(\mathbf{G}, \mathbf{H}); \mathbf{G}, \mathbf{H}) \geq u_i(a_i, \mathbf{a}_{-i}^*(\mathbf{G}, \mathbf{H}); \mathbf{G}, \mathbf{H}), \quad \forall a_i \in A_i, \forall i \in \mathcal{N} \tag{4}$$

where $\mathbf{a}_{-i}^*(\mathbf{G}, \mathbf{H})$ is the Nash action profile of players other than i. Denote the (first stage) reduced form utility function of player i by:

$$U_i(\mathbf{G}, \mathbf{H}) = u_i(a_i^*(\mathbf{G}, \mathbf{H}), \mathbf{a}_{-i}^*(\mathbf{G}, \mathbf{H}), \mathbf{G}, \mathbf{H}) \tag{5}$$

Let $c \geq 0$ denote the cost to a player of a bilateral link in $\mathbf{G}$. We assume that all players have the same cost of link formation in $\mathbf{G}$. Since $\mathbf{H}$ is assumed to be historically given, any linking costs incurred in the formation of $\mathbf{H}$ are assumed to be sunk. We will show later that our results continue to hold if players who are more connected in $\mathbf{H}$ have lower costs of forming links in $\mathbf{G}$. The *net* utility of player i in the network $\mathbf{G}$ is given by:

$$U_i(\mathbf{G}, \mathbf{H}) - d_i(\mathbf{G}) c \tag{6}$$

Note that this is also player i's payoff from the two-stage game.

Pairwise Stable Equilibrium

Following Goyal and Joshi (2006) we will say that $\mathbf{G}$ is a *pairwise stable equilibrium* (or *pws-equilibrium*) if: (i) no player has an incentive to delete any subset of own links, and (ii) if $ij \notin \mathbf{G}$, then these two players who are unlinked in $\mathbf{G}$ cannot profit by establishing a link:

$$U_i(\mathbf{G}+ij,\mathbf{H})-U_i(\mathbf{G},\mathbf{H}) > c \;\Rightarrow\; U_j(\mathbf{G}+ij,\mathbf{H})-U_j(\mathbf{G},\mathbf{H}) < c \tag{7}$$

The Process of Network Formation

We specify an infinitely repeated sequential game of network formation in $\mathbf{G}$ in the first stage to select from the set of pws-equilibria. Players are indexed according to their degree in network $\mathbf{H}$, with player 1 (respectively, N) having the lowest (respectively, highest) degree. The process starts from the empty network, $\mathbf{G}(0) \equiv \mathbf{G}^e$. Players move sequentially in the order of their index to effect changes to the network. The player chosen to move in stage t is called the *active* player. Suppose the active player is i and the network in place when i has to move is $\mathbf{G}(t-1)$, $t \in \{1,2,..\}$. The active player i observes the current network $\mathbf{G}(t-1)$ and makes two decisions: (i) delete any subset of own links, and (ii) propose a link to player j, $ij \notin \mathbf{G}(t-1)$, who provides i with the largest incremental utility from an additional link (if there are two or more such players, then propose a link to the player who has the highest index).[6] Player j, to whom the link is proposed, is the *passive* player whose reactive role is simply to reciprocate or reject the proposed link ij. The link ij is formed only if player j acquiesces. Any unilateral deletions of own links by the active player, and the possible formation of a new link with a passive player, leads to a network $\mathbf{G}(t)$. We will say in this case that $\mathbf{G}(t)$ is *reachable* from $\mathbf{G}(t-1)$. After all players have moved in this way, the sequential process is repeated. We state that a *round* is completed after players 1 through N have moved; thus, if player i is the active player after r rounds have been completed, then the current network confronting i is $\mathbf{G}(rN+i-1)$. The number of rounds is open-ended. However, we will see that a pws-equilibrium is reached in a *finite* number of rounds.

Improving Paths and Cycles

Following Jackson and Watts (2002), a sequence of networks, $\{\mathbf{G}(0),\mathbf{G}(1),..,\mathbf{G}(t),..\}$ is an *improving path*[7] if the following three conditions hold: (i) $\mathbf{G}(t)$ is reachable from $\mathbf{G}(t-1)$, $t \in \{1,2,..\}$; (ii) if the active player is i when the current network is $\mathbf{G}(t-1)$, and $\mathbf{G}(t) = \mathbf{G}(t-1)+ij$, then:

$$U_i(\mathbf{G}(t),\mathbf{H})-d_i(\mathbf{G}(t))c \;>\; U_i(\mathbf{G}(t-1),\mathbf{H})-d_i(\mathbf{G}(t-1))c \tag{8}$$

$$U_j(\mathbf{G}(t),\mathbf{H})-d_j(\mathbf{G}(t))c \;\geq\; U_j(\mathbf{G}(t-1),\mathbf{H})-d_j(\mathbf{G}(t-1))c \tag{9}$$

and, (iii) if the active player is i when the current network is $\mathbf{G}(t-1)$, and $\mathbf{G}(t) \subseteq \mathbf{G}(t-1)$, then:

$$U_i(\mathbf{G}(t),\mathbf{H})-d_i(\mathbf{G}(t))c \geq U_i(\mathbf{G}(t-1),\mathbf{H})-d_i(\mathbf{G}(t-1))c \tag{10}$$

[6]Within each round we can specify that the active player first deletes links and then proposes a new link. We will see that in our framework this specification of order of play is inconsequential.

[7]In Jackson and Watts (2002), such a path is called a *simultaneous* improving path, to refer to the fact that a change in the network entails both deletion and addition of links. For the sake of brevity we simply refer to it as an improving path.

Note that our notion of an improving path assumes myopic behavior on part of the players. The active-passive tuple of players in any stage do not take into account the consequences of their actions for the players that will follow later.[8] We will say that a set $\mathcal{G}$ of networks is a *cycle* if for any $\mathbf{G}, \mathbf{G}' \in \mathcal{G}$, there exists an improving path from $\mathbf{G}$ to $\mathbf{G}'$, and a cycle $\mathcal{G}$ is *closed* if no element of $\mathcal{G}$ is on an improving path to a network $\mathbf{G}' \notin \mathcal{G}$. A simple adaptation of Jackson and Watts (2002, Lemma 1) establishes that *an improving path leads to a pws-equilibrium or a closed cycle.*

3 The strategic complementarities model

We assume that the utility function is given by (3) where the actions of players within each network are strategic complements and $\lambda > 0$ is sufficiently small. In addition, we assume that each player i's actions x_i and y_i across the two networks are also (weak) strategic complements, i.e. $\gamma \in (0,1)$. We begin by analyzing the second stage Nash game in actions $a_i = (x_i, y_i) \in A_i$ on $\mathbf{G}$ and $\mathbf{H}$. It is easily verified that the Nash actions are non-zero and characterized by the first order conditions:

$$\mathbf{x} = \mathbf{1} + \lambda \mathbf{G}\mathbf{x} + \gamma \mathbf{y}, \quad \mathbf{y} = \mathbf{1} + \psi \mathbf{H}\mathbf{y} + \gamma \mathbf{x} \tag{11}$$

Assuming $\psi\mu_{\max}(\mathbf{H}) < 1$ so that $\mathbf{M}(\mathbf{H},\psi) \equiv [m_{ij}] = [\mathbf{I} - \psi\mathbf{H}]^{-1}$ exists and is non-negative, it follows that $\mathbf{y} = \mathbf{b}(\mathbf{H},\psi,\mathbf{1}) + \gamma\mathbf{M}(\mathbf{H},\psi)\mathbf{x}$. Note that if $\gamma = 0$, then the two networks are independent and, as in Ballester et al. (2006, Theorem 1), a player's action in a given network is proportional to the player's KB centrality in that network only and unrelated to the player's KB centrality in the other network. Therefore, the assumption that $\gamma \neq 0$ strategically connects actions across the two network and makes them interdependent. To formally capture these interdependencies, it is convenient to define the notion of a composite network $\mathbf{L}$ as $\mathbf{L} \equiv [l_{ij}] = \lambda\mathbf{G} + \gamma^2\mathbf{M}(\mathbf{H},\psi)$. Observe that the composite network tracks the degree of players in $\mathbf{G}$ and their total number of walks in $\mathbf{H}$.

Proposition 1 *If $\mu_{\max}(\mathbf{L}) < 1$, then the optimal Nash equilibrium actions are*:

$$\mathbf{x}^*(\mathbf{G},\mathbf{H}) = [\mathbf{I} - \mathbf{L}]^{-1}\boldsymbol{\alpha}_{\mathbf{H}} = \mathbf{b}(\mathbf{L},1,\boldsymbol{\alpha}_{\mathbf{H}}) \tag{12}$$

$$\mathbf{y}^*(\mathbf{G},\mathbf{H}) = \mathbf{M}(\mathbf{H},\psi)(\mathbf{1} + \gamma\mathbf{b}(\mathbf{L},1,\boldsymbol{\alpha}_{\mathbf{H}})) = \mathbf{b}(\mathbf{H},\psi,\boldsymbol{\beta}_{\mathbf{L}}) \tag{13}$$

where $\boldsymbol{\alpha}_{\mathbf{H}} = \gamma\mathbf{b}(\mathbf{H},\psi,\mathbf{1}) + \mathbf{1}$ and $\boldsymbol{\beta}_{\mathbf{L}} = \gamma\mathbf{b}(\mathbf{L},1,\boldsymbol{\alpha}_{\mathbf{H}}) + \mathbf{1}$.

The Nash equilibrium level of action $x_i^*(\mathbf{G},\mathbf{H})$ is determined by player i's $\boldsymbol{\alpha}_{\mathbf{H}}$-weighted KB centrality in network $\mathbf{L}$, while $y_i^*(\mathbf{G},\mathbf{H})$ is determined by player i's $\boldsymbol{\beta}_{\mathbf{L}}$-weighted KB centrality in network $\mathbf{H}$. To simplify notation we will write the Nash equilibrium actions as x_i^* and y_i^* and drop the reference to $\mathbf{G}$ and $\mathbf{H}$ when there is no ambiguity. To get some perspective on Nash action levels chosen by distinct players i

[8] Such a formulation is fairly standard (Jackson and Watts 2002, p. 273) and allows tractability as compared to a far-sighted network formation approach (Dutta et al. 2005). It would hold for example when players highly discount the future.

and j, note from (13) that:

$$y_i^* - y_j^* = [b_i(\mathbf{H}, \psi, \mathbf{1}) - b_j(\mathbf{H}, \psi, \mathbf{1})] + \gamma \sum_{k=1}^{N} [m_{ik} - m_{jk}] \, x_k^* \tag{14}$$

Therefore, if player i has greater KB centrality than j in network $\mathbf{H}$ (i.e. $b_i(\mathbf{H}, \psi, \mathbf{1}) > b_j(\mathbf{H}, \psi, \mathbf{1})$), then compared to j there are more walks from i to every other player (i.e. $m_{ik} > m_{jk}$ for all k). Accordingly, i's action y_i^* in $\mathbf{H}$ is greater than action y_j^* of j. The actions x_i^* and x_j^* are derived from the composite matrix $\mathbf{L}$ and therefore depend on both the neighborhood in $\mathbf{G}$ and number of walks in $\mathbf{H}$. Letting $\mathbf{M}^s(\mathbf{H}, \psi) = \left[m_{ij}^{[s]}\right]$:

$$x_i^* - x_j^* = \lambda \sum_{s=1}^{\infty} \left[s\gamma^{2(s-1)} \sum_{q=1}^{N} \left\{ \sum_{l=1}^{N} (g_{il} - g_{jl}) \, m_{lq}^{[s-1]} + \gamma^{2s} \left(m_{iq}^{[s]} - m_{jq}^{[s]} \right) \right\} \boldsymbol{\alpha}_{\mathbf{H}q} \right] \tag{15}$$

Therefore, x_i^* is greater than x_j^* if player i has greater KB centrality than j in $\mathbf{H}$ ($m_{iq}^{[s]} > m_{jq}^{[s]}$ for all s) and an augmented neighborhood in $\mathbf{G}$ that includes that of j.

Substituting the Nash equilibrium action levels into the utility function yields the first stage reduced form utility of player i:

$$\begin{aligned} U_i(\mathbf{G}, \mathbf{H}) &= \frac{1}{2} \left[y_i^*(\mathbf{G}, \mathbf{H}) - x_i^*(\mathbf{G}, \mathbf{H}) \right]^2 + (1-\gamma) x_i^*(\mathbf{G}, \mathbf{H}) y_i^*(\mathbf{G}, \mathbf{H}) && (16) \\ &= \frac{1}{2} \left[b_i(\mathbf{H}, \psi, \mathbf{1}) + \sum_{j=1}^{N} \widehat{m}_{ij} x_j^*(\mathbf{G}, \mathbf{H}) \right]^2 + (1-\gamma) x_i^*(\mathbf{G}, \mathbf{H}) y_i^*(\mathbf{G}, \mathbf{H}) && (17) \end{aligned}$$

where $\gamma\mathbf{M} - \mathbf{I} \equiv [\widehat{m}_{ij}]$. Therefore, the reduced utility of player i is strictly increasing in i's KB centrality in $\mathbf{H}$, the Nash action levels of i in both networks, as well as the Nash actions x_j^* in $\mathbf{G}$ of all players other than i. The next two lemmas are based on these properties of the reduced utility function.

Lemma 1 *Consider networks* $\mathbf{G}$ *and* $\mathbf{H}$. *Suppose that* $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$, *and* $\mathbf{N}_k(\mathbf{G}) \subseteq \overline{\mathbf{N}}_j(\mathbf{G}) = \mathbf{N}_j(\mathbf{G}) \cup \{j\}$. *The reduced utility function satisfies*:

$$\begin{aligned} U_i(\mathbf{G}+ij, \mathbf{H}) - U_i(\mathbf{G}, \mathbf{H}) &\geq U_i(\mathbf{G}+ik, \mathbf{H}) - U_i(\mathbf{G}, \mathbf{H}) && (18) \\ U_j(\mathbf{G}+ij, \mathbf{H}) - U_j(\mathbf{G}, \mathbf{H}) &\geq U_k(\mathbf{G}+ik, \mathbf{H}) - U_k(\mathbf{G}, \mathbf{H}) && (19) \\ U_j(\mathbf{G}+jk, \mathbf{H}) - U_j(\mathbf{G}, \mathbf{H}) &\geq U_k(\mathbf{G}+jk, \mathbf{H}) - U_k(\mathbf{G}, \mathbf{H}) && (20) \end{aligned}$$

The inequalities in (18)-(20) are strict if $b_j(\mathbf{H}, \psi, \mathbf{1}) > b_k(\mathbf{H}, \psi, \mathbf{1})$ *and/or* $\mathbf{N}_k(\mathbf{G}) \subset \mathbf{N}_j(\mathbf{G})$.

In Lemma 1, inequality (18) shows that player i gains more from linking with player j than with player k, if j is more central in $\mathbf{H}$ than k, and j's augmented neighborhood in $\mathbf{G}$ contains the neighborhood of k. The following two inequalities (19)-(20) show that the incremental utility of player j is greater than that of player k when both form a link with the same partner or with each other. The intuition behind these results is based on the fact that the reduced utility of player i is increasing in Nash action levels $\left(x_i^*, x_{-i}^*\right)$ and y_i^*. The Nash actions (x_i^*, y_i^*) increase to a greater extent when i forms a link with j rather than k. Further, x_{-i}^* is greater when j rather than k links with i.

Lemma 2 *Let* $\mathbf{H}$ *be a connected network.*

(a) *If* $\mathbf{G} \subset \mathbf{G}'$ *and* $ij \notin \mathbf{G}'$*, then:*

$$U_i(\mathbf{G}' + ij, \mathbf{H}) - U_i(\mathbf{G}', \mathbf{H}) > U_i(\mathbf{G} + ij, \mathbf{H}) - U_i(\mathbf{G}, \mathbf{H}) > 0 \tag{21}$$

(b) *If* $\mathbf{H} \subset \mathbf{H}'$ *and* $ij \notin \mathbf{G}$*, then:*

$$U_i(\mathbf{G} + ij,\ \mathbf{H}') - U_i(\mathbf{G},\ \mathbf{H}') > U_i(\mathbf{G} + ij,\ \mathbf{H}) - U_i(\mathbf{G},\ \mathbf{H}) \tag{22}$$

Lemma 2 examines the incentives of players to connect in denser networks and allows us to take into account link formation costs for the active and passive player. Part (a) in particular implies that players do not have an incentive to delete existing links along an improving path. If existing links are always maintained, then it is not possible to have a closed cycle on an improving path. It follows from Jackson and Watts (2002, Lemma 1) *that an improving path will converge to a pws-equilibrium network. In particular, a pws-equilibrium exists.* Our main result characterizes the limit network $\mathbf{G}$ when $\mathbf{H}$ is a connected NSG as well as when $\mathbf{H}$ is a KB-regular graph.

Theorem 1 *Suppose* $\mathbf{H}$ *is a connected network and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *converges to a limit network* $\mathbf{G}$*.*

(a) *If* $\mathbf{H}$ *is a connected NSG, then* $\mathbf{G}$ *is empty, complete or has at most one non-singleton NSG component.*

(b) *If* $\mathbf{H}$ *is a KB-regular graph, then* $\mathbf{G}$ *is either empty or complete.*

We begin with the case where $\mathbf{H}$ is a connected NSG. It is worth noting that if $\gamma = 0$, then following König et al. (2014), the equilibrium architecture of $\mathbf{G}$ is a NSG. [9] However, since the two networks are independent under $\gamma = 0$, there is no relationship between the centrality of a player in the two networks. By assuming $\gamma > 0$, our results show that the centrality of players in the equilibrium network is driven by their centrality in the inherited network. Recall that players are indexed according to their degree in $\mathbf{H}$. Given that $\mathbf{H}$ is a NSG, the degree partition $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}), P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$ coincides with the partition of players according to their KB-centralities in $\mathbf{H}$ (König et al., 2014, Proposition 1). Since $\mathbf{H}$ is

[9] While a NSG architecture may seem to be intuitive in the context of strategic complementarities, they can also arise under strategic substitutes. This is shown in a recent paper by Kinateder and Merlino (2017) who study public goods on networks under heterogeneous costs and values.

connected, $P_0(\mathbf{H}) = \emptyset$. We prove part (a) of Theorem 1 by first describing how players in the highest and lowest echelons of $\mathcal{P}(\mathbf{H})$ fare in the limit network $\mathbf{G}$. We let $\mathcal{P}(\mathbf{G}) = \{P_0(\mathbf{G}),\ P_1(\mathbf{G}), ..., P_n(\mathbf{G})\}$ and refer to networks on an improving path between $\mathbf{G}(0) \equiv \mathbf{G}^e$ and the limit (pws-equilibrium) network as *interim* networks.

Proposition 2 *Suppose* $\mathbf{H}$ *is a connected NSG,* $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$*, and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. Then*:

(*a*) $P_m(\mathbf{H}) \subseteq P_n(\mathbf{G})$ *and each player in* $P_m(\mathbf{H})$ *is linked to all players in the network* $\mathbf{G}$*.*

(*b*) $P_1(\mathbf{H}) \subseteq P_1(\mathbf{G})$ *and* $\mathbf{N}_i(\mathbf{G}) = P_n(\mathbf{G})\ \forall i \in P_1(\mathbf{G})$*.*

We begin by recording the following corollary to Lemmas 1 and 2:

Corollary 1: *Suppose that* $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$ *in* $\mathbf{H}$ *and* $\mathbf{N}_k(\mathbf{G}) \subseteq \overline{\mathbf{N}}_j(\mathbf{G})$ *in* $\mathbf{G}$*. If players* k *and* i *have an incentive to form a link in the network* $\mathbf{G}$*, then players* j *and* i *also have an incentive to form a link in the network* $\mathbf{G}'$*, where* $\mathbf{G} \subseteq \mathbf{G}'$*.*

We now describe the intuition behind Proposition 2, which also provides some insight into the mechanics of network formation and how the NSG architecture of $\mathbf{H}$ induces preferential attachment in $\mathbf{G}$. Consider round 1 and the active player $1 \in P_1(\mathbf{H})$ who begins the network formation process from $\mathbf{G}(0) \equiv \mathbf{G}^e$. There are no links to delete. Player 1 will propose a link to player N. The reason for this is as follows. For each $i \in \mathcal{N}\backslash\{1, N\}$, since $\mathbf{N}_i(\mathbf{G}^e) = \emptyset \subset \overline{\mathbf{N}}_N(\mathbf{G}^e) = \{N\}$ and $b_N(\mathbf{H}, \psi, \mathbf{1}) \geq b_i(\mathbf{H}, \psi, \mathbf{1})$, it follows from (18) that $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) \geq U_1(\mathbf{G}^e + 1i, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H})$, i.e. the highest incremental utility to player 1 accrues from a link with player $N \in P_m(\mathbf{H})$. The assumption that $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$ ensures that this link is profitable. Since $b_N(\mathbf{H}, \psi, \mathbf{1}) > b_1(\mathbf{H}, \psi, \mathbf{1})$ and $\mathbf{N}_1(\mathbf{G}^e) = \emptyset \subset \overline{\mathbf{N}}_N(\mathbf{G}^e) = \{N\}$, it follows from (20) that $U_N(\mathbf{G}^e + 1N, \mathbf{H}) - U_N(\mathbf{G}^e, \mathbf{H}) > U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$, i.e. player N will reciprocate the link. Therefore the improving path from $\mathbf{G}^e$ leads to the interim network $\mathbf{G}(1) \equiv \mathbf{G}^e + 1N$.

We now claim that each active player $k \in \{2, 3, ..., N-1\}$ in round 1 will also find it most profitable to propose a link to player N and player N will reciprocate. Suppose this is true for player $k \in \{2, 3, ..., N-2\}$ leading to the network $\mathbf{G}(k)$. We show that this is also true for active player $k+1$. Note that $b_{k+1}(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$. Further, in the network $\mathbf{G}(k-1)$, $\mathbf{N}_k(\mathbf{G}(k-1)) = \emptyset \subset \overline{\mathbf{N}}_{k+1}(\mathbf{G}(k-1)) = \{k+1\}$. Since players k and N could form a mutually profitable link when the network was $\mathbf{G}(k-1)$, and $\mathbf{G}(k-1) \subset \mathbf{G}(k)$, it follows from Corollary 1 that players $k+1$ and N have an incentive to form a link in the network $\mathbf{G}(k)$. It remains to show that player N is the most profitable player with whom player $k+1$ can form a link. Note that for all $j \in \{1, 2, ..., k\}$, $\mathbf{N}_j(\mathbf{G}(k)) = \{N\} \subset \overline{\mathbf{N}}_N(\mathbf{G}(k)) = \{1, 2, ..., k, N\}$, while for all $j \in \{k+2, ..., N-1\}$, $\mathbf{N}_j(\mathbf{G}(k)) = \emptyset \subset \overline{\mathbf{N}}_N(\mathbf{G}(k))$. Since $b_N(\mathbf{H}, \psi, \mathbf{1}) \geq b_j(\mathbf{H}, \psi, \mathbf{1})$ for all j, it follows from (18) that:

$$U_{k+1}(\mathbf{G}(k) + (k+1)N, \mathbf{H}) - U_{k+1}(\mathbf{G}(k), \mathbf{H}) \geq U_{k+1}(\mathbf{G}(k) + (k+1)j, \mathbf{H}) - U_{k+1}(\mathbf{G}(k), \mathbf{H}), \quad \forall j \in \mathcal{N}\backslash\{k+1, N\}$$

establishing that the most profitable link for player $k+1$ is with player N. This establishes the induction result. Therefore, at the end of stage $N-1$, player N is linked to all $N-1$ players. In stage N, it follows from (21) that the active player N will not delete any links and has no additional links to propose. Therefore player $N \in P_n(\mathbf{G})$. The following rounds successively connect players in $P_m(\mathbf{H})$ in decreasing order of their index to all players in $\mathbf{G}$ and thus $P_m(\mathbf{H}) \subseteq P_n(\mathbf{G})$. This establishes Proposition 2(a).

Note that players in $P_1(\mathbf{H})$ are connected in $\mathbf{G}$ to players in $P_m(\mathbf{H})$ under the assumption that $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$. In fact, players in $P_1(\mathbf{H})$ will be connected in $\mathbf{G}$ to, and only to, the maximally connected players in $\mathbf{G}$. This is because of the manner in which interim networks from $\mathbf{G}(0) \equiv \mathbf{G}^e$ evolve under strategic complementarities. In particular, if $\mathbf{G}(rN)$ is the interim network at the end of round r, then $\mathbf{N}_k(\mathbf{G}(rN)) \subseteq \overline{\mathbf{N}}_j(\mathbf{G}(rN))$ if $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$. This is because the mutual profitability of the link ik during round r ensures the mutual profitability of the link ij as well. This observation leads to the following:

Corollary 2: *Suppose that $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$ in $\mathbf{H}$. If players k and i have an incentive to form a link in an interim network $\mathbf{G}'$ on an improving path, then players j and i have an incentive to form a link in an interim network $\mathbf{G}''$, where $\mathbf{G}' \subseteq \mathbf{G}''$.*

Now suppose player $i \notin P_n(\mathbf{G})$ is linked to player $k \in P_1(\mathbf{H})$ in some interim network. From Corollary 2, player i and every other player $j \in \mathcal{N}\backslash\{i,k\}$ will also find it mutually profitable to form a link along an improving path since by definition of the set $P_1(\mathbf{H})$ we have $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$. But then player i should be maximally linked in $\mathbf{G}$, contradicting $i \notin P_n(\mathbf{G})$. Therefore, all players in $P_1(\mathbf{G})$ can only be linked to maximally linked players in $\mathbf{G}$ thereby establishing Proposition 2(b).

At this point we digress briefly to consider the extreme cases of empty and complete networks as candidates for a pws-equilibrium. It is immediately obvious that for sufficiently low (respectively, high) costs of link formation the pws-equilibrium network will be complete (respectively, empty). Recalling the indexation of the players, suppose that for the given network $\mathbf{H}$, the linking cost c satisfies:

$$0 \leq c < U_1(\mathbf{G}^e + 12, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H})$$

i.e. the least central player in $\mathbf{H}$ finds it profitable to forge a link with the second least central player in $\mathbf{H}$ when no connections exist among the players in the network $\mathbf{G}$. It follows from Corollary 2 that each player can form a mutually profitable link with the remaining $N-1$ players. Therefore, this is a sufficient condition under which the improving path will lead to the complete network $\mathbf{G}^c$. Next we provide a sufficient condition for the empty network. Suppose that for the given network $\mathbf{H}$:

$$U_{N-1}(\mathbf{G}^e + (N-1)N, \mathbf{H}) - U_{N-1}(\mathbf{G}^e, \mathbf{H}) < c$$

i.e. at least one of the two most central players in $\mathbf{H}$ has no incentive to form a link in the empty network. It follows that each active player from stage 1 onwards will not have an incentive to announce a link with a partner since no link is mutually profitable. Thus the improving path leads to the empty network.[10] We now turn to those players who occupy the intermediate sets in $\mathcal{P}(\mathbf{H})$. Note from Corollary 2 that if $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$ *in* $\mathbf{H}$, (equivalently, $d_j(\mathbf{H}) \geq d_k(\mathbf{H})$ given $\mathbf{H}$ is NSG), then $d_j(\mathbf{G}) \geq d_k(\mathbf{G})$. We will say more about the neighborhoods of players j and k after identifying an important "coarsening" property of the equilibrium degree partition.

Proposition 3 (***Non-Partial Overlap Property***) *Suppose* $\mathbf{H}$ *is a connected NSG and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the limit network* $\mathbf{G}$*. The degree partition* $\mathcal{P}(\mathbf{G})$ *displays the following property: For each* $l = 1, 2, ..., n-1$*, let* $R_l = \{s', s'+1, ..., s''\}$ *be the set of indices such that* $P_s(\mathbf{H}) \cap P_l(\mathbf{G}) \neq \emptyset$ *for* $s \in R_l$*. Then* $P_l(\mathbf{G}) = \cup_{s \in R_l} P_s(\mathbf{H})$*.*

The non-partial overlap property can be construed as an "equal treatment in equilibrium" property: two players with the same KB centrality in network $\mathbf{H}$ cannot be treated differently in the equilibrium network $\mathbf{G}$; it is possible however for two players with different KB centralities in $\mathbf{H}$ to be treated the same in the equilibrium network $\mathbf{G}$. As an illustration, consider the NSG in figure 2. Any set of players who have the same degree in $\mathbf{H}$, for instance players 1 and 2, also have the same degree in $\mathbf{G}$. An important implication of this proposition is that *the cardinality of the degree partition of* $\mathbf{G}$ *cannot exceed that of* $\mathbf{H}$. In other words, $\mathbf{G}$ inherits a weakly "coarser" degree partition from $\mathbf{H}$. In figure 2, we can have a coarsening "at the bottom" when the lower elements in the degree partition of $\mathbf{H}$ are merged in $\mathbf{G}$, or a coarsening "at the top" when the upper elements in the degree partition of $\mathbf{H}$ are combined in $\mathbf{G}$.

The non-partial overlap property implies that $\mathbf{G}$ will inherit a NSG architecture (please see the appendix for the proof). Intuitively, since the degree partition of $\mathbf{G}$ is composed either of individual elements of $\mathcal{P}(\mathbf{H})$, or union of elements of $\mathcal{P}(\mathbf{H})$, it follows that $\mathcal{P}(\mathbf{G})$ exhibits the nested neighborhood structure that is characteristic of a NSG. If there is more than one non-singleton component, then from each component we can pick a player whose KB centrality in $\mathbf{H}$ is at least as high as any other player in that component. The two players picked in this manner have a mutually profitable link from Corollary 1.

The coarsening property reveals two different effects in equilibrium. On one hand, we find that those with the highest (respectively, lowest) KB centrality in $\mathbf{H}$ will have the highest (respectively, lowest) KB centrality in $\mathbf{G}$. Hence we see a kind of "*silver spoon*" effect in equilibrium suggesting that preferential attachment is a wide-spread phenomenon. As long as there are strategic complementarities, nodes with high centrality in one network will continue to be highly central in the other network. On the other hand, we also see a "*silver lining*" effect. When $\mathbf{G}$ inherits a "coarser" degree partition from $\mathbf{H}$, there is clearly the possibility of "limited" mobility, i.e. players in certain degree partitions in $\mathbf{H}$ can be in the same or higher degree partitions of $\mathbf{G}$. This is largely driven by costs of link formation in $\mathbf{G}$ and we focus more on this issue of inequality in networks in the next section. Given that $\mathbf{H}$ is a NSG, there is another very useful

[10] Of course there is a potential coordination problem here. It is quite possible that in some non-empty (denser) network the incremental utilities for some players exceed c and consequently mutually profitable links are possible. However, since players respond myopically to the existing network, the improving path remains "trapped" at $\mathbf{G}^e$.

consequence. As shown by König et al. (2014), in a NSG, KB centrality coincides with degree centrality. So the above statements about the silver spoon effect and limited mobility can be based on node degrees.

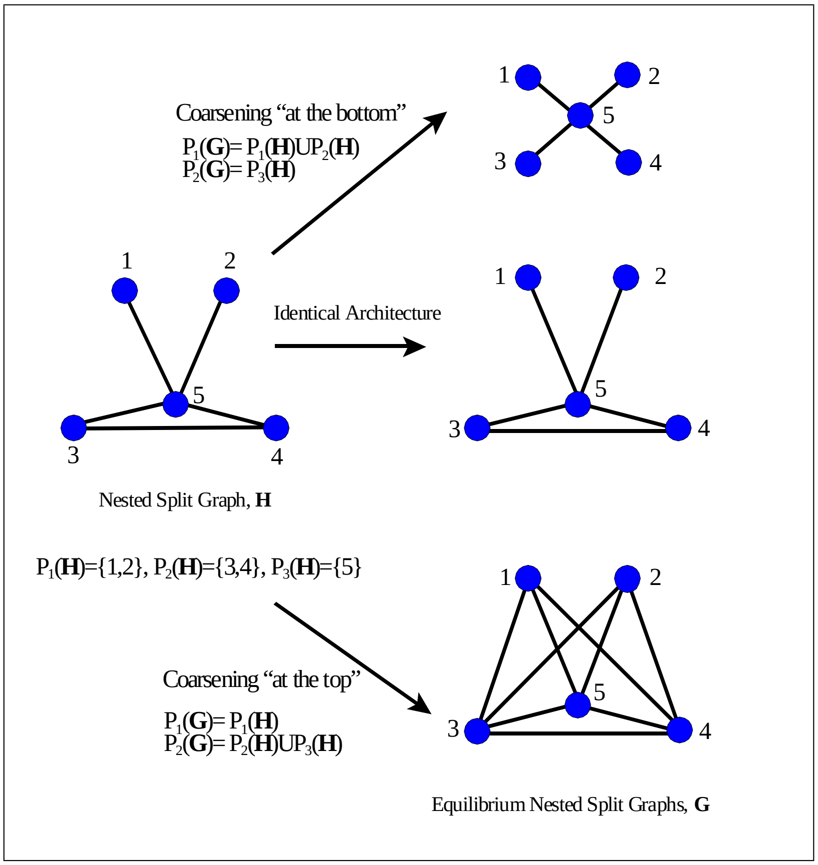

Figure 2: Non-Partial Overlap Property

Now consider part (b) of Theorem 1. Let $\mathbf{H}$ be a KB-regular graph. Consider round 1 and the active player $1 \in P_1(\mathbf{H})$ in stage 1. Without loss of generality, we can focus on player N, since all players are equally central in $\mathbf{H}$. If $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) \leq c$, then no link will be proposed by player 1. Since all players are ex-ante identical, no links are formed subsequently and the improving path ends in $\mathbf{G}^e$. If $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$, then the link $1N$ is formed. Subsequently each active player $k \in \{2, 3, ..., N-1\}$ in round 1 will propose a link to player N and player N will reciprocate. Therefore player N is maximally connected at the end of round 1. From Lemma 2(a), each subsequent round ends with a player getting maximally connected. The improving path therefore leads to the complete network.

Example 1: Let $\mathbf{H}$ be the NSG network in Figure 1(c). Fix $\lambda = \psi = 0.101$ and $c = 0.2263$. Invertibility of $\mathbf{L}$ requires $\gamma \in (0, 0.368)$. The following architectures arise in equilibrium as γ increases: (i) For $0 < \gamma \leq 0.181$, $\mathbf{G} = \mathbf{G}^e$; (ii) For $0.181 < \gamma \leq 0.183$, $\mathbf{G}$ consists of two groups where one is an isolated set of players $\{1, 2, 3, 4, 5\}$ and the other is a star with player 9 as its center; (iii) For $0.183 < \gamma \leq 0.197$, $\mathbf{G}$ assumes a 4-dominant group structure where players $\{1, 2, 3, 4, 5\}$ are isolated and $\{6, 7, 8, 9\}$ form a complete component; (iv) For $0.197 < \gamma \leq 0.204$, $\mathbf{G} = \mathbf{H}$; and (v) For $0.204 < \gamma \leq 0.368$, $\mathbf{G} = \mathbf{G}^c$. Therefore, as γ increases, the first links in equilibrium networks are formed with the player having the highest KB centrality in $\mathbf{H}$, and progressive coarsening "at the top" culminates in the complete network. ■

Next, we elaborate on some aspects of our framework.

Coarsening of Degree Partition in the Equilibrium Network

The fact that the cardinality of the degree partition of $\mathbf{G}$ cannot exceed that of $\mathbf{H}$ could be of practical significance in many instances. This is because the coarsening property reduces the number of architectures that have to considered as candidates for equilibrium. For example, if $\mathbf{H}$ is a star or an "interlinked star" as in figure 1(a) and 1(b), then there is no NSG with a coarser degree partition that respects the non-partial overlap property. Therefore, if a pws equilibrium $\mathbf{G}$ other than the empty or complete exists for some intermediate level of linking costs, then $\mathbf{G}$ must have the same architecture as $\mathbf{H}$.

Cost of Link Formation

While we have assumed identical costs of link formation across agents, our results for the case where $\mathbf{H}$ is a NSG will only be reinforced if a player's link formation cost is decreasing in their KB-centrality in $\mathbf{H}$.[11] For instance, suppose that $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$, and therefore $c_j \leq c_k$. Additionally suppose that in the given network $\mathbf{G}$, $\mathbf{N}_k(\mathbf{G}) \subseteq \overline{\mathbf{N}}_j(\mathbf{G})$, and that player k has a profitable link with some i. From Lemma 1 it follows that player j also has a profitable link with i:

$$U_j(\mathbf{G} + ij, \mathbf{H}) - U_j(\mathbf{G}, \mathbf{H}) \geq U_k(\mathbf{G} + ik, \mathbf{H}) - U_k(\mathbf{G}, \mathbf{H}) > c_k \geq c_j$$

Therefore, having link formation costs decline with KB-centrality strengthens preferential attachment and incentivizes more connected players in $\mathbf{H}$ to also form more links in $\mathbf{G}$.

The Network Formation Process

Our network formation process starts from $\mathbf{G}^e$ and has players moving sequentially in increasing order of their KB centrality in $\mathbf{H}$. The order in which the players move is not important for the final result as long as the active player i in each stage forms a link with the player who provides i with the highest incremental utility from the link. The fact that the dynamic starts from $\mathbf{G}^e$ is, however, important for our results. It ensures that the sequential process proceeds in a specific way with more central players in $\mathbf{H}$ getting connected first. If the process starts at some *arbitrary* $\mathbf{G}$ in which a player with high KB-centrality in $\mathbf{H}$ has few links in $\mathbf{G}$, then it would not be possible to characterize the limit network. Given the non-linearity in reduced utility due to feedback loops in both networks $\mathbf{G}$ and $\mathbf{H}$, it would be difficult to determine who the active player in any given stage of the link formation game should connect to.

Relationship between Degree, KB Centrality and Net Utility

The focus of our paper is on KB centrality, or how a player's influence in one network affects their centrality in another. Given that in equilibrium, if $\mathbf{H}$ is a NSG, then so is $\mathbf{G}$, following König et al. (2014) this

[11] We thank an anonymous referee for noting that the cost of forming a (business) link in $\mathbf{G}$ could be lower for two players who are already connected (socially) in $\mathbf{H}$.

translates into equivalence in degrees as well. The same is obviously true for regular networks. So we find that players who are more central in one network will also be more central in the other network, allowing for coarsening determined by costs of link formation. However, this equivalence does not always carry over to *net* utilities. It is obvious that net utilities and KB centrality will be proportional to each other if $\mathbf{G}$ is either empty or complete, since utilities will now be determined by $\mathbf{H}$. This is also true for the parameters in Example 1. More generally, this proportionality will hold when the net utilities of the players forming a link exceeds the indirect gross utility of players not involved in this link. We now identify a sufficient condition under which this is true. Define the following threshold utilities:

$$\begin{aligned}\overline{u} &= U_k\left(\mathbf{G}^c, \mathbf{H}^c\right) - U_k\left(\mathbf{G}^c - ij, \mathbf{H}^c\right), \quad k \in \mathcal{N}\backslash\{i,j\} \\ \underline{u} &= U_k\left(\mathbf{G}^e + ik, \mathbf{H}^e\right) - U_k\left(\mathbf{G}^e, \mathbf{H}^e\right) - c, \quad k \in \mathcal{N}\backslash\{i\}\end{aligned}$$

Note that these threshold utilities are independent of the identities of the players i, j, k. Here, $\overline{u}$ is the largest *indirect gross* utility that can accrue to a player not involved in the link, while $\underline{u}$ is the smallest *direct net* utility that can ensue to a player from forming a link. If $\underline{u} \geq \overline{u}$, then net utility is proportional to KB centrality in an equilibrium NSG network.

Proposition 4 *Suppose $\underline{u} \geq \overline{u}$. If $\mathbf{G}$ is a pws-equilibrium network that is a NSG, then for $i \in P_{l'}(\mathbf{G})$ and $j \in P_l(\mathbf{G})$, where $l' > l$:*

$$U_i\left(\mathbf{G}, \mathbf{H}\right) - d_i\left(\mathbf{G}\right)c > U_j\left(\mathbf{G}, \mathbf{H}\right) - d_j\left(\mathbf{G}\right)c$$

4 Network Inequality

We have seen that when $\mathbf{H}$ is a NSG, then the network $\mathbf{G}$ assumes a NSG architecture with a possibly coarser degree partition. We will see that the degree distribution in $\mathbf{G}$ is strongly impacted by the cost of link formation in $\mathbf{G}$. For this purpose we will fix a reference network $\mathbf{G}^*$ that is identical to $\mathbf{H}$ and establish a set of baseline incremental utilities with respect to $\mathbf{G}^*$. Recalling that $\mathbf{H}$ (and therefore $\mathbf{G}^*$) has the degree partition $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}),\, P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$, define the following incremental utilities:

$$\begin{aligned}\underline{\delta}_l &= U_i(\mathbf{G}^* + ik, \mathbf{H}) - U_i(\mathbf{G}^*, \mathbf{H}), \quad i \in P_l(\mathbf{H}),\ k \in P_{m-l}(\mathbf{H}),\ l = 1, 2, ..., \lfloor m/2 \rfloor - 1 \\ \overline{\delta}_l &= U_i(\mathbf{G}^*, \mathbf{H}) - U_i(\mathbf{G}^* - ij, \mathbf{H}), \quad i \in P_l(\mathbf{H}),\ j \in P_{m-l+1}(\mathbf{H}),\ l = 1, 2, ..., \lfloor m/2 \rfloor\end{aligned}$$

Note that $\underline{\delta}_l$ and $\overline{\delta}_l$ depend only on the relevant elements of the partition $\mathcal{P}(\mathbf{H})$ and not on the identity of the players who comprise these partitions. Let:

$$\underline{\delta} = \max\left\{\underline{\delta}_l, l = 1, 2, ..., \lfloor m/2 \rfloor - 1\ \right\}, \quad \overline{\delta} = \ \min\left\{\overline{\delta}_l, l = 1, 2, ..., \lfloor m/2 \rfloor\right\}$$

If $\underline{\delta} < \overline{\delta}$, then cost of link formation $c \in (\underline{\delta}, \overline{\delta})$ can support $\mathbf{G}^*$ as a pws-equilibrium network. For this case:

$$\underline{\delta}_l < c, \quad l = 1, 2, ..., \lfloor m/2 \rfloor - 1 \tag{23}$$

$$c < \overline{\delta}_l, \ l = 1, 2, ..., \lfloor m/2 \rfloor \tag{24}$$

The first condition (23) ensures that a player in $P_l(\mathbf{G}^*)$ has no incentive to propose a link to a player in $P_{m-l}(\mathbf{G}^*)$. From (18) in Lemma 1 it follows that players in $P_l(\mathbf{G}^*)$ have no incentive to propose links to less central players in $P_{m-l-1}(\mathbf{G}^*), P_{m-l-2}(\mathbf{G}^*), ..., P_{l+1}(\mathbf{G}^*)$ etc., either. The second condition (24) ensures that a player in $P_l(\mathbf{G}^*)$ has no incentive to delete links with the least central of its partners in $\mathbf{H}$. It then follows from (18) that players in $P_l(\mathbf{G}^*)$ have no incentive to delete any existing links. *Therefore, for* $c \in (\underline{\delta}, \overline{\delta})$, *there is a pws-equilibrium network that mirrors the degree distribution of* $\mathbf{H}$.[12]

Before proceeding, we need to establish a measure that can compare the variation in degree distribution across two networks. There are numerous ways in which inequality in degree can be measured, and here we choose a particularly simple measure that looks at "first degree dominance" in the degree distribution of the players. We will say that the degree distribution of $\mathbf{G}$ *first order dominates* the degree distribution of $\mathbf{G}'$ if $d_i(\mathbf{G}) \geq d_i(\mathbf{G}')$, $\forall i \in \mathcal{N}$ with strict inequality for at least one player i. We will denote this formally as $\mathcal{D}(\mathbf{G}) \succeq_{FOD} \mathcal{D}(\mathbf{G}')$.

Proposition 5 *Suppose* $\mathbf{H}$ *is a connected NSG network and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to some non-empty network* $\mathbf{G}$. *If* $\overline{\delta} \leq \overline{\delta}_1 < c$, *then* $\mathcal{D}(\mathbf{G}^*) \succeq_{FOD} \mathcal{D}(\mathbf{G})$. *If* $0 < c < \underline{\delta}_1 \leq \underline{\delta}$, *and* $\mathbf{G}^*$ *lies on the improving path from* $\mathbf{G}^e$, *then* $\mathcal{D}(\mathbf{G}) \succeq_{FOD} \mathcal{D}(\mathbf{G}^*)$.

To prove this result, suppose linking costs are high relative to the levels required to support $\mathbf{G}^*$ as a pws-equilibrium. In particular, suppose that $\overline{\delta} \leq \overline{\delta}_1 < c$. Note that $\mathbf{G}^*$ cannot be an interim network on the improving path leading to $\mathbf{G}$. Since $\mathbf{G}^*$ includes all links between minimally and maximally connected players, if $\mathbf{G}^*$ was on the improving path, then in some interim network $\mathbf{G}' \subset \mathbf{G}^* - ij$ the active player $i \in P_1(\mathbf{H})$ would have formed a profitable link with $j \in P_m(\mathbf{H})$. However, by virtue of Lemma 2(a):

$$U_i(\mathbf{G}' + ij, \mathbf{H}) - U_i(\mathbf{G}', \mathbf{H}) < U_i(\mathbf{G}^*, \mathbf{H}) - U_i(\mathbf{G}^* - ij, \mathbf{H}) = \overline{\delta}_1 < c \tag{25}$$

contradicting the profitability of the link.

Next, note that the limiting network $\mathbf{G} \subset \mathbf{G}^*$, otherwise there is at least one link in $\mathbf{G}$ that does not exist in $\mathbf{G}^*$. Let kl be the *first* link that is formed along the improving path that does not exist in $\mathbf{G}^*$ and let $\mathbf{G}'$ be the interim network when kl is established. Then, by the choice of the link, $\mathbf{G}' \subseteq \mathbf{G}^*$. Now let $c' \in (\underline{\delta}, \overline{\delta})$ denote any cost supporting $\mathbf{G}^*$ as a pws-equilibrium and note that $c' < \overline{\delta} \leq \overline{\delta}_1 < c$. From Lemma 2(a):

$$U_k(\mathbf{G}^* + kl, \mathbf{H}) - U_k(\mathbf{G}^*, \mathbf{H}) > U_k(\mathbf{G}' + kl, \mathbf{H}) - U_k(\mathbf{G}', \mathbf{H}) \geq c > c'$$

and identically for player l. Therefore players kl have a mutually profitable link in $\mathbf{G}^*$ for $c' \in (\underline{\delta}, \overline{\delta})$ contradicting its pws-equilibrium property.

[12] This is not to say that there will always be an improving path leading to $\mathbf{G}^*$. Since players behave myopically along an improving path, it is possible that the process will lead to a pws-equilibrium network different from $\mathbf{G}^*$.

The limiting network $\mathbf{G}$ must be less dense than $\mathbf{G}^*$. Now applying the same reasoning as in (25), it follows that $\mathbf{G}$ does not have any links of the form ij, $i \in P_1(\mathbf{H})$ and $j \in P_m(\mathbf{H})$, i.e. links between the minimally and maximally connected players. From Lemma 1, equation (18), any player who has at least one link must be linked to players in $P_m(\mathbf{H})$. It follows that $d_i(\mathbf{G}) \leq d_i(\mathbf{G}^*)$ $\forall i \in \mathcal{N}$ and $d_i(\mathbf{G}) < d_i(\mathbf{G}^*)$ $\forall i \in P_1(\mathbf{H}) \cup P_m(\mathbf{H})$. The players at the lower extreme of the degree distribution of $\mathbf{G}$ see a strict decrease in their number of connections (in fact they are now isolated in $\mathbf{G}$) while the players at the upper extreme continue to be maximally connected (though to a smaller set of players). To the extent that players at the lower end of the degree distribution are left worse off than those at the upper end, the first order domination of $\mathbf{G}$ by $\mathbf{G}^*$ can be construed as a move towards greater inequality in degree. *Therefore, an increase in linking costs above $\overline{\delta}_1$ increases the inequality in degree distribution relative to $\mathbf{G}^*$.*

Now consider the case where $c < \underline{\delta}_1$. The assumption that $\mathbf{G}^*$ lies on the improving path leading to $\mathbf{G}$ is needed to address the coordination problems created by strategic complementarity. Since the initial interim networks are less dense than $\mathbf{G}^*$, they may preclude the formation of profitable links that exist in $\mathbf{G}^*$. At any interim network $\mathbf{G}' \supset \mathbf{G}^*$, the active player $i \in P_1(\mathbf{H})$ profits by announcing a link with $k \in P_{m-1}(\mathbf{H})$:

$$U_i(\mathbf{G}' + ik, \mathbf{H}) - U_i(\mathbf{G}', \mathbf{H}) > U_i(\mathbf{G}^* + ik, \mathbf{H}) - U_i(\mathbf{G}^*, \mathbf{H}) = \underline{\delta}_1 > c$$

From Lemma 1, equation (20), player k will accept. It follows that the limiting network $\mathbf{G}$ will contain all links of the form ik, $i \in P_1(\mathbf{H})$ and $k \in P_{m-1}(\mathbf{H})$. Therefore $d_i(\mathbf{G}) \geq d_i(\mathbf{G}^*)$ $\forall i \in \mathcal{N}$ and $d_i(\mathbf{G}) > d_i(\mathbf{G}^*)$ $\forall i \in P_1(\mathbf{H}) \cup P_{m-1}(\mathbf{H})$. *To the extent that players at the lower end of the degree distribution are now better linked, a reduction in linking costs below $\underline{\delta}_1$ can be interpreted as decreasing the inequality in degree distribution relative to $\mathbf{G}^*$.*

Example 2: Consider Example 1 with $\lambda = \psi = 0.101$ and $\gamma = 0.2$. We now examine the impact of costs on degree distribution and obtain the following equilibrium configurations: (i) For $0 \leq c \leq 0.2254$: $\mathbf{G} = \mathbf{G}^c$; (ii) For $0.2255 \leq c \leq 0.2307$: $\mathbf{G} = \mathbf{H}$; (iii) For $0.2308 \leq c \leq 0.2450$: $\mathbf{G}$ assumes a 4-dominant group structure where agents {1, 2, 3, 4, 5} are isolated and {6, 7, 8, 9} form a complete component; (iv) For $0.2451 \leq c \leq 0.2488$: $\mathbf{G}$ consists of two groups where one is an isolated group consisting of players {1, 2, 3, 4, 5} and the other group is a star with player 9 as its center; and (v) For $c > 0.2488$: $\mathbf{G} = \mathbf{G}^e$, the empty network. Therefore, as costs increase, we lose the silver lining aspect as the number of links in $\mathbf{G}$ decrease culminating in $\mathbf{G}^e$. ■

5 Application: Overlapping generations model

Numerous studies have shown that family background significantly influences intergenerational mobility and inequality (Becker and Tomes (1979, 1986) and Corak (2013). In this section we provide an application of our multigraph analysis to the intergenerational transmission of inequaility via a simple overlapping generations model.[13] In the model, time is discrete and indexed as $t \in \{0, 1, 2, ...\}$. The number of *families* is fixed and indexed as $i \in \{1, 2, ..., N\}$. An agent born in period t in family i is indexed as $i(t)$ and is said to belong to generation t. The set of agents $\{i(0) : i = 1, 2, ..., N\}$ in generation 0, and their business network $\mathbf{G}_{(0)}$, is historically given. Agents live for two periods. Each agent $i(t-1)$ of generation $t-1$ procreates exactly one agent $i(t)$ in the beginning of period $t \in \{1, 2, ...\}$. Agents $i(t)$ of each generation $t \in \{1, 2, ...\}$ establish a business network $\mathbf{G}_{(t)}$ in the first period of their lives. In the second period of their lives they procreate agent $i(t+1)$ and then interact with generation $t+1$ in the network $\mathbf{H}_{(t+1)}$ to communicate valuable professional information. Therefore, generation t interacts in network $\mathbf{G}$ with their own cohort, and in network $\mathbf{H}$ with their parent's cohort. The business links of generation t are inherited as the social connections of generation $t+1$, i.e. $\mathbf{H}_{(t+1)} = \mathbf{G}_{(t)}$. Contingent on the inherited $\mathbf{G}_{(t)}$, agents $i(t+1)$ of generation $t+1$ establish their business network $\mathbf{G}_{(t+1)}$ and then exert efforts in both networks.[14] In the social network $\mathbf{H}_{(t+1)}$, agent $i(t+1)$ invests effort y_i to glean information from the business contacts of the parent that can be leveraged in the business realm (with net payoff to generation t from this information-providing activity is normalized to zero). Agent $i(t+1)$ is induced to exert more effort in $\mathbf{H}_{(t+1)}$ if a social connection $j(t+1)$ is doing so, i.e. y_i and y_j are strategic complements in $\mathbf{H}_{(t+1)}$. In the business network $\mathbf{G}_{(t+1)}$, agent $i(t+1)$ exerts effort x_i in some non-cooperative game of strategic complementarities with other agents of generation $t+1$. Finally, the efforts of an agent $i(t+1)$ across the two networks are strategic complements: increasing y_i generates more profitable business know-how and thus induces more effort in the business network. The utility function of agent $i(t)$ is given by (3) for each $i(t) \in \{1, 2, ..., N\}$ and $t \in \{0, 1, 2, ...\}$.

We will assume that $\mathbf{G}_{(0)}$ is a connected NSG so that the overlapping generation starts with an asymmetric distribution of links. We know from Proposition 3 that the degree partition of each $\mathbf{G}_{(t)}$ is a coarsening of $\mathbf{G}_{(t-1)}$ for $t \in \{1, 2, ...\}$. We are interested in characterizing the limit network, if it exists. The following result establishes monotonicity in the coarsening of the degree partition of the business network of each succesive generation. Given that the set of networks is finite, this monotonicity implies that a limit network exists.

Proposition 6 *For each* $t \in \{1, 2, ...\}$*:*
(*a*) *If* $\mathcal{D}\left(\mathbf{G}_{(0)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(1)}\right)$*, then* $\mathcal{D}\left(\mathbf{G}_{(t)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(t+1)}\right)$*.*
(*b*) *If* $\mathcal{D}\left(\mathbf{G}_{(1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(0)}\right)$*, then* $\mathcal{D}\left(\mathbf{G}_{(t+1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(t)}\right)$*.*

Therefore, if $\mathcal{D}\left(\mathbf{G}_{(1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(0)}\right)$, then the limit network could be complete, while if $\mathcal{D}\left(\mathbf{G}_{(0)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(1)}\right)$, then the limit network could be empty. We now identify sufficient conditions for the limit network to be empty or complete. We also identify a sufficient condition under which the limit network remains connected. Define the following threshold levels of cost of link formation:

$$U_i(\mathbf{G^e} + ij,\ \mathbf{H}^e) - U_i(\mathbf{G^e},\ \mathbf{H}^e) = \underline{c},\ U_i(\mathbf{G^e} + ij,\ \mathbf{H}^c) - U_i(\mathbf{G^e},\ \mathbf{H}^c) = \overline{c}$$

[13] We would like to thank the associate editor and an anonymous referee for suggesting this overlapping generations application of the multigraph model.

[14] For analytical tractability, we once again suppose that each generation is "myopic" in the sense that it does not take into account how their link formation in the professional sphere will impact subsequent generations.

$\underline{c}$ (respectively, $\bar{c}$) is the cost at which any two players $i, j \in \mathcal{N}$ are indifferent between forming a link or remaining unconnected when $\mathbf{G}$ is empty and $\mathbf{H}$ is empty (respectively, complete). Note that these threshold cost levels are independent of i and j and thus apply across all players. Now consider a star network $\mathbf{H}^s$ where i_p is a peripheral player and i_s is center of the star, and let c_s denote the cost that leaves i_p indifferent between connecting to i_s or remaining isolated:

$$U_{i_p}(\mathbf{G^e} + i_p i_s,\ \mathbf{H}^s) - U_{i_p}(\mathbf{G^e},\ \mathbf{H}^s) = c_s$$

Since $\mathbf{H}^e \subset \mathbf{H}^s \subset \mathbf{H}^c$, it follows from Lemma 2(b) that $\bar{c} > c_s > \underline{c}$. We can now obtain the following result:

Proposition 7 *Suppose $\mathbf{G}_{(0)}$ is a connected NSG and c is the cost of link formation. If $c \leq c_s$, then the limit network is connected. Additionally:*

(a) If $c \leq \underline{c}$, then the limit network is complete.

(b) If $c \geq \bar{c}$, then the limit network is empty.

It is reasonable to ask under what conditions is the limit network non-empty for $c \in (c_s, \bar{c})$. For $c \in (c_s, \bar{c})$, suppose there exists a k-dominant group network $\mathbf{H}^{d_k}$, $k \in \{3, 4, .., N-1\}$, such that for any two players i and j who belong to the dominant group:

$$U_i(\mathbf{G^e} + ij,\ \mathbf{H}^{d_k}) - U_i(\mathbf{G^e},\ \mathbf{H}^{d_k}) \geq c, \quad U_j(\mathbf{G^e} + ij,\ \mathbf{H}^{d_k}) - U_j(\mathbf{G^e},\ \mathbf{H}^{d_k}) \geq c$$

and for any two players i and j belonging to a $(k-1)$-dominant group network $\mathbf{H}^{d_{k-1}}$:

$$U_i(\mathbf{G^e} + ij,\ \mathbf{H}^{d_{k-1}}) - U_i(\mathbf{G^e},\ \mathbf{H}^{d_{k-1}}) < c, \quad U_j(\mathbf{G^e} + ij,\ \mathbf{H}^{d_{k-1}}) - U_j(\mathbf{G^e},\ \mathbf{H}^{d_{k-1}}) < c$$

Such a k-dominant group network $\mathbf{H}^{d_k}$ for the given cost c of link formation is called a *c-self-sustainable core* (or c-SSC). For a given c, if there exists a c-SSC $\mathbf{H}^{d_k} \subseteq \mathbf{G}_{(0)}$, then the limit network is non-empty. This follows from the observation that the k players who constitute the c-SSC will have an incentive to link in each $\mathbf{G}_{(t)}$ and thus the limit network cannot be empty.

6 Strategic substitutability in actions

We will now explore strategic substitutability in actions along two dimensions. We will begin with the case where actions x_i and y_i of each agent i are strategic substitutes. We will then consider the case where $\{x_1, x_2, ..., x_N\}$ are strategic substitutes in the business network $\mathbf{G}$.

6.1 Strategic substitutability in effort across networks

We will specify the utility of player i as in (3) but with $\gamma \in (-1, 0)$. Studying equilibrium network formation is difficult since $\gamma < 0$, creates a tradeoff in action choice across networks. However, because of this tradeoff we can now focus on understanding how players choose their optimal actions, an aspect that was not possible under $\gamma > 0$ where actions across networks serve to reinforce each other. We start by analyzing the optimal action choice. From the first order conditions, for γ sufficiently small, we have:

$$x_i^*(\mathbf{G},\mathbf{H}) = \sum_{q=1}^{N}\sum_{s=0}^{\infty} l_{iq}^s \alpha_{Hq} \approx \sum_{q=1}^{N}\sum_{s=0}^{\infty} (\lambda g_{iq})^s \alpha_{Hq} = \sum_{q=1}^{n} w_{iq}\alpha_{Hq} \tag{26}$$

$$y_i^*(\mathbf{G},\mathbf{H}) = b_i(\mathbf{H},\psi,\mathbf{1}) + \gamma \sum_{q=1}^{N} m_{iq} x_q^*(\mathbf{G},\mathbf{H}) \tag{27}$$

where $\mathbf{W} = [w_{ij}] = [\mathbf{I}-\lambda\mathbf{G}]^{-1}$. Note that y_i^* depends on $b_i(\mathbf{H},\psi\mathbf{1})$ and a higher $b_i(\mathbf{H},\psi,\mathbf{1})$ implies a higher y_i^*. On the other hand, in the expression for y_i^*, the impact of x_q^* is negative because $\gamma < 0$. We can now state our first result which highlights the fact that under strategic substitutes there is a tradeoff between KB centrality in $\mathbf{H}$ and neighborhood size in $\mathbf{G}$.

Proposition 8 *Let $\gamma \in (-1,0)$. If $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$ and $\overline{\mathbf{N}}_k(\mathbf{G}) \supseteq \mathbf{N}_j(\mathbf{G})$, then:*

$$x_i^*(\mathbf{G}+ij,\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H}) \leq x_i^*(\mathbf{G}+ik,\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H}) \tag{28}$$

$$x_j^*(\mathbf{G}+ij,\mathbf{H}) - x_j^*(\mathbf{G},\mathbf{H}) \leq x_k^*(\mathbf{G}+ik,\mathbf{H}) - x_k^*(\mathbf{G},\mathbf{H}) \tag{29}$$

Observe that Proposition 8 reverses the conditions on the attractiveness of a link partner as compared to the case of strategic complementarities. Ceteris paribus, linking to player k rather than to player j increases x_i^* by a greater extent when the augmented neighborhood of player k subsumes that of j in $\mathbf{G}$. Now, taking into account the negative externality arising from network $\mathbf{H}$ due to $\gamma < 0$, the depressive effect on x_i^* is smaller when the partner is less central in $\mathbf{H}$. The net effect makes player k more attractive as a partner for i than player j. Similar reasoning yields (29). Given the conditions on centrality and neighborhood size, player k will choose a higher action level in $\mathbf{G}$ than player j.

Studying equilibrium network formation under strategic substitutability is a non-trivial matter and can allow for a multiplicity of solutions including corner solutions. Therefore our goal in this section will be to primarily identify new equilibrium network types. For simplicity, instead of a connected NSG we will now assume that network $\mathbf{H}$ is a k-dominant group network, $\mathbf{H}^{d_k}$, and the payoff function satisfies some regularity properties.[15] Consider a dominant group structure where there are two groups in $\mathbf{H}^{d_k}$ and therefore two KB measures. The connected group, $P_1\left(\mathbf{H}^{d_k}\right)$, has KB measure $b_1(\mathbf{H},\psi,\mathbf{1}) > 0$ while the isolated group, $P_0\left(\mathbf{H}^{d_k}\right)$, has KB measure 0. Letting $i, i' \in P_1\left(\mathbf{H}^{d_k}\right)$ and $j, j' \in P_0\left(\mathbf{H}^{d_k}\right)$, the following possible architectures shown in figure 3 can arise in equilibrium.

[15] The advantage of assuming that $\mathbf{H}$ is a k-dominant group network is that it it minimizes spillovers while still allowing us to identify interesting architectures. Moreover, we also assume that *(i)* $\frac{\Delta x_i^*(\mathbf{G^e}+ij',\mathbf{H^D})}{\Delta x_j^*(\mathbf{G^c}-ij',\mathbf{H^D})} \leq \frac{\gamma m_{jj}}{\gamma m_{ii}}$; *(ii)* $(1+\gamma m_{jj}) > (1+\gamma m_{ii}) > 0$; *and (iii)* $\frac{\Delta x_i^*(\mathbf{G^e}+ii',\mathbf{H^D}) - \Delta x_i^*(\mathbf{G^e}+ij,\mathbf{H^D})}{\Delta x_{i'}^*(\mathbf{G^c}-ii',\mathbf{H^D})} \geq \frac{-\gamma m_{ii'}}{(1-\gamma m_{ii})}$. These are mostly technical assumptions which ensure that the payoff functions are well behaved.

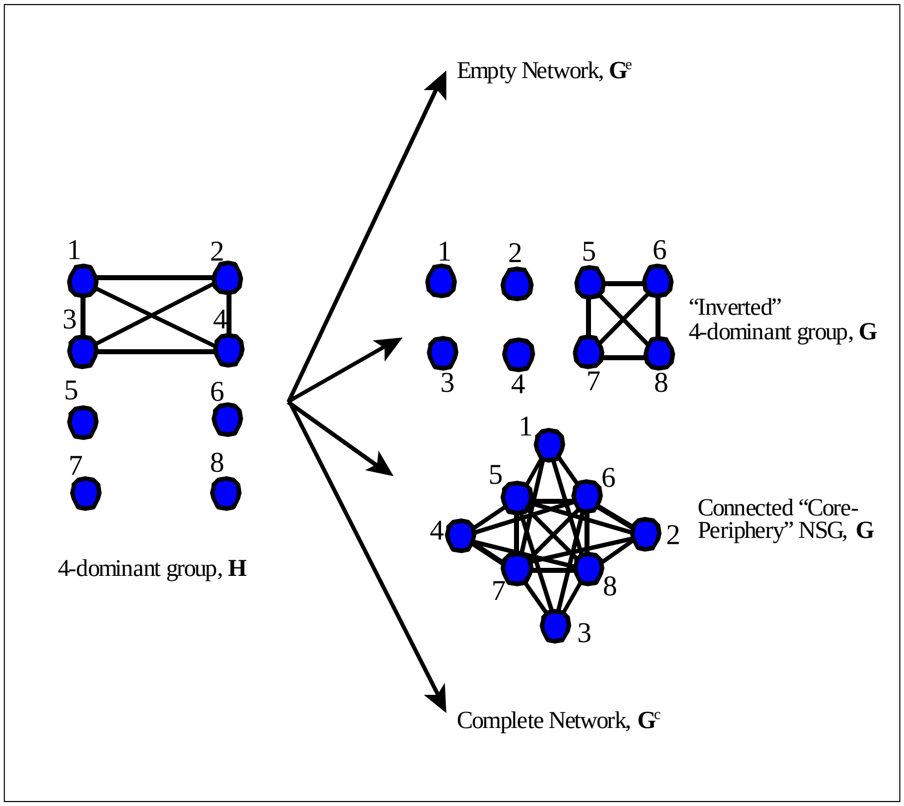

Figure 3: Strategic Substitutability in Efforts Across Networks

- Empty, if $U_j(\mathbf{G^e}+jj', \mathbf{H}^{d_k}) - U_j(\mathbf{G^e},\ \mathbf{H}^{d_k}) < c$.
- Complete, if $U_j(\mathbf{G^e}+jj', \mathbf{H}^{d_k}) - U_j(\mathbf{G^e},\ \mathbf{H}^{d_k}) > c$, and the reduced utility to players in $P_1\left(\mathbf{H}^{d_k}\right)$ is convex in own links. Otherwise, the condition for the complete network is, $U_i(\mathbf{G^c}, \mathbf{H}^{d_k}) - U_i(\mathbf{G^c} - ii', \mathbf{H}^{d_k}) > c$.
- An "inverted" dominant group network $\mathbf{G}^{d_k}$ where $P_0\left(\mathbf{H}^{d_k}\right) = P_1\left(\mathbf{G}^{d_k}\right)$ and $P_1\left(\mathbf{H}^{d_k}\right) = P_0\left(\mathbf{G}^{d_k}\right)$. This is possible if:

$$U_j(\mathbf{G}^{d_k}+ij', \mathbf{H}^{d_k}) - U_j(\mathbf{G}^{d_k}, \mathbf{H}^{d_k}) < c < U_j(\mathbf{G^e}+jj', \mathbf{H}^{d_k}) - U_j(\mathbf{G^e}, \mathbf{H}^{d_k})$$

  This condition states that the isolated group is willing to form links with another member from the isolated group but not with a dominant group member. Consequently, all links are formed among the isolated group members, and another dominant group emerges where the roles are reversed. This mirrors the notion of a leisure class à la Veblen where those who inherit a dominant group do not form any links in $\mathbf{G}$ and exert no effort in that network. Those who inherit the empty network on the other hand form a connected group. For this situation to arise, we need that, starting from $\mathbf{G^e}$, the cost of link formation should be less than the benefits from a link between two players in $P_0\left(\mathbf{H}^{d_k}\right)$. At the same time, given $\gamma < 0$, linking to a player in $P_1\left(\mathbf{H}^{d_k}\right)$ creates a negative externality whose magnitude makes the link unprofitable. This leads to a network $\mathbf{G}$ where connections are the opposite to those in $\mathbf{H}$.

- A "core-periphery" NSG network $\mathbf{G^{CP}}$ which displays the same inverted architecture as the previous case with $P_0\left(\mathbf{H}^{d_k}\right) = P_2\left(\mathbf{G^{CP}}\right)$ and $P_1\left(\mathbf{H}^{d_k}\right) = P_1\left(\mathbf{G^{CP}}\right)$. The core, $P_2\left(\mathbf{G^{CP}}\right)$, is fully connected while the periphery, $P_1\left(\mathbf{G^{CP}}\right)$, is connected to the core but not to each other. This is possible if:

$$\begin{array}{rcl} U_j(\mathbf{G^e}+jj', \mathbf{H}^{d_k}) - U_j(\mathbf{G^e}, \mathbf{H}^{d_k}) & > & c \\ U_i(\mathbf{G^{CP}}+ii', \mathbf{H}^{d_k}) - U_i(\mathbf{G^{CP}}, \mathbf{H}^{d_k}) & < & c < U_i(\mathbf{G}^{d_k}+ij', \mathbf{H}^{d_k}) - U_i(\mathbf{G}^{d_k}, \mathbf{H}^{d_k}) \end{array}$$

  The first inequality and convexity in own links (from the assumed regularity properties) lead to all members of the isolated group in $\mathbf{H}^{d_k}$ forming links among themselves resulting first in a dominant group. Given this dominant group network, the second inequality states that players in $P_1\left(\mathbf{H}^{d_k}\right)$ have no incentive to connect to each other but have an incentive to connect to those in $P_0\left(\mathbf{H}^{d_k}\right)$. Given that agents in $P_0\left(\mathbf{H}^{d_k}\right)$ are isolated in $\mathbf{H}$, they do not incur any negative externalities and will therefore reciprocate the link. The intuition is similar to the previous case except that costs have to be even lower to enable those in $P_1\left(\mathbf{G^{CP}}\right)$ to form links.

6.2 Strategic substitutability in actions within networks

We have so far assumed that actions of the N players within each network are strategic complements. We will now assume that actions in $\mathbf{G}$ are strategic substitutes (i.e. $\lambda < 0$). We will continue to maintain that actions x_i and y_i of each agent i are strategic complements. Since $\mathbf{H}$ is the inherited network, it makes little sense to assume that actions in $\mathbf{H}$ are strategic substitutes ($\psi < 0$). Hence we will continue to assume that actions in $\mathbf{H}$ are strategic complements (i.e. $\psi > 0$). Our main result is the following:

Proposition 9 *Suppose the utility function is given by (3) with $\psi > 0$, $\lambda < 0$ and $\gamma > 0$. For any network $\mathbf{H}$, the improving path from $\mathbf{G}(0) \equiv \mathbf{G}^e$ leads to the empty network.*

Therefore, when efforts in the business network $\mathbf{G}$ are strategic substitutes, then no links will be formed in equilibrium. When $\lambda < 0$, links exert negative externality in $\mathbf{G}$, i.e. the formation of a link reduces utility. Consequently no links are formed. It is worth noting that the structure of the inherited network has no impact on the formed network.

Remark: While we do not consider the case where actions are strategic substitutes in $\mathbf{H}$ (i.e. $\psi < 0$), it can be shown that if $\lambda > 0$, and $\mathbf{G}$ and $\mathbf{H}$ are NSG, then KB-centrality will be inverted in the two networks: players with higher KB-centrality in $\mathbf{H}$ have lower KB-centrality in $\mathbf{G}$.

7 Conclusion

In this paper we examine interactions across two networks to determine its impact on endogenous network formation. We assume that players inherit one network and form the other network, and then choose actions in both. We study the implications of strategic complementarity and substitutability in actions across both networks. We use this to study the evolution of inherited features like inequality through a simple overlapping generations model. Our paper opens up a number of new directions for future research. Industrial organization is one such area with several possibilities. This multiple network approach can be used to analyze multimarket competition and collaborative R&D networks where efforts across networks may be substitutes or complements. This can also provide a framework for the growing empirical literature on these topics. Another important area for both theoretical and empirical work is development economics where individuals are simultaneously involved in multiple relationships like trade, finance, favors and advice. Possibly the most important research question relates to multiplex/multigraph network formation. Ever since the seminal papers of Jackson and Wolinsky (1996) and Bala and Goyal (2000), a wide range of theoretical models of network formation have been proposed to address different types of network situations. However, none of these models deal with network formation while taking the interplay between multiple networks into account. We believe that our paper is only a first step and future research will study the simultaneous formation of multiple networks while taking strategic interactions across them into account.

8 Appendix

We begin with the following technical result. Let $\mathbf{M}^s = \mathbf{M} \times \mathbf{M} \times \cdots \times \mathbf{M}$ (s times).

Lemma 3 *Suppose* $\mathbf{H}$ *is NSG and let* $\mathbf{M}^s(\mathbf{H},\xi) \equiv \left[m_{ij}^{[s]}\right]$, $s \in \mathbb{Z}_+$. *If* $i \in P_l(\mathbf{H})$ *and* $j \in P_{l'}(\mathbf{H})$ *where* $l' > l$, *then* $m_{ik}^{[s]} < m_{jk}^{[s]}$ $\forall k \in \mathcal{N}$, $\forall s \in \mathbb{Z}_+$. *In particular, for any* $N \times 1$ *vector* $\mathbf{a}$, $\sum_{k=1}^{N} m_{ik}^{[s]} a_k < \sum_{k=1}^{N} m_{jk}^{[s]} a_k$.

Proof: We will first establish that $m_{ik} < m_{jk}$. Consider all walks from i and j to $k \in \mathcal{N}$. For each walk $\{k_1, k_2, .., k_l\}$ connecting i to k, there is a corresponding walk of the same length $\{k_1, k_2, .., k_l\}$ connecting j to k since $k_1 \in \mathbf{N}_i(\mathbf{H}) \subset \mathbf{N}_j(\mathbf{H})$. We now show that there are strictly more walks from j to k than from i to k. Suppose $k \in \mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$. Then there is a walk of length 1 from j to k but no corresponding walk from i to k. Now suppose $k \in \mathbf{N}_i(\mathbf{H})$. There are 3 cases, and in each case j has walks to k for which i does not have corresponding walks to k. (i) $l < l' \leq \lfloor m/2 \rfloor$. All neighbors of i and j belong to elements of the degree partition with index greater than $\lfloor m/2 \rfloor$ and are interconnected. There are $d_i(\mathbf{H}) - 1$ walks of length 2 from i to k and $d_j(\mathbf{H}) - 1$ walks of length 2 from j to k. Therefore there are $d_j(\mathbf{H}) - d_i(\mathbf{H})$ additional walks of length 2 from j to k through neighbors in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$; (ii) $l \leq \lfloor m/2 \rfloor < l'$. In $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$, consider those players whose degree is greater than that of i. These players also have k as a neighbor and therefore j has additional walks of length 2 to k through these players. Now consider those players in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$ whose degree is less than that of i. If any of these has k as a neighbor, then once again j has an additional walk of length 2 to k. If any of these players, say l, does not have k as a neighbor, then j has an additional walk of length three through $\{l, j\}$. (iii) $l > \lfloor m/2 \rfloor$. The players in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$ have lower degree than i. For each such player l there is a walk of length three through $\{l, j\}$, but no corresponding walk of length 3 from i to k. This establishes $m_{ik} < m_{jk}$. It follows that:

$$m_{ik}^{[2]} = \sum_{l=1}^{N} m_{il} m_{lk} < \sum_{l=1}^{N} m_{jl} m_{lk} = m_{jk}^{[2]}$$

It now follows inductively that $m_{ik}^{[s]} = \sum_{l=1}^{N} m_{il}^{[s-1]} m_{lk} < \sum_{l=1}^{N} m_{jl}^{[s-1]} m_{lk} = m_{jk}^{[s]}$, $\forall s \in \mathbb{Z}_+$. $\blacksquare$

Proof of Proposition 1: The first order conditions for $i \in \mathcal{N}$ are given by:

$$\begin{aligned}\frac{\partial u_i}{\partial x_i} &= 1 - x_i + \lambda \sum_{j=1}^{N} g_{ik} x_j + \gamma y_i = 0 \\ \frac{\partial u_i}{\partial y_i} &= 1 - y_i + \psi \sum_{k=1}^{N} h_{ik} y_k + \gamma x_i = 0\end{aligned}$$

Clearly $(x_i, y_i) = (0, 0)$ for all $i \in \mathcal{N}$ is not a Nash equilibrium. Writing the pair of first order conditions in matrix form and solving yields the result. $\blacksquare$

Before proving Lemmas (1) and (2) we establish a few preliminary results. Since the links in the network $\mathbf{H}$ remain fixed and only those in the network $\mathbf{G}$ are allowed to change, it will be convenient to simplify (16) further to reflect this fact. Letting $\gamma \mathbf{M} - \mathbf{I} \equiv [\widehat{m}_{ij}]$, it follows from the first order condition that:

$$y_i^* - x_i^* = b_i(\mathbf{H}, \psi, \mathbf{1}) + \sum_{j=1}^{N} \widehat{m}_{ij} b_j \left(\mathbf{L}, 1, \boldsymbol{\alpha}_{\mathbf{H}}\right) \equiv \Phi_i(\mathbf{G}, \mathbf{H}) \tag{30}$$

Recall that $\mathbf{b}\left(\mathbf{L}, 1, \boldsymbol{\alpha}_{\mathbf{H}}\right) = \left(\mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \cdots\right) \boldsymbol{\alpha}_{\mathbf{H}}$. Let $\mathbf{L}^s = \left[l_{ij}^{[s]}\right]$ and $\boldsymbol{\alpha}_{\mathbf{H}q} = b_q(\mathbf{H}, \psi, \mathbf{1}) + 1$ denote the q^{th} row of the vector $\boldsymbol{\alpha}_{\mathbf{H}}$. Substituting into (30) yields:

$$\Phi_i(\mathbf{G}, \mathbf{H}) = b_i(\mathbf{H}, \psi, \mathbf{1}) + \sum_{j=1}^{N} \widehat{m}_{ij} \left(\sum_{s=0}^{\infty} \sum_{q=1}^{N} l_{jq}^{[s]} \boldsymbol{\alpha}_{\mathbf{H}q} \right) \tag{31}$$

Written in this way, only the elements of $[l_{ij}^{[s]}]$ will be influenced by the formation of links in the network $\mathbf{G}$ and we can observe the role that centrality of players in network $\mathbf{H}$ will play in this regard. The incremental utility of player i from adding the link ii' in the network $\mathbf{G}$ is:

$$U_i(\mathbf{G} + ii', \mathbf{H}) - U_i(\mathbf{G}, \mathbf{H}) = \frac{1}{2}\left[\Phi_i(\mathbf{G} + ii', \mathbf{H}) + \Phi_i(\mathbf{G}, \mathbf{H})\right]\left[\Phi_i(\mathbf{G} + ii', \mathbf{H}) - \Phi_i(\mathbf{G}, \mathbf{H})\right] + (1 - \gamma)\Delta(ii')$$

where $\Delta(ii') \equiv x_i^*(\mathbf{G} + ii', \mathbf{H}) y_i^*(\mathbf{G} + ii', \mathbf{H}) - x_i^*(\mathbf{G}, \mathbf{H}) y_i^*(\mathbf{G}, \mathbf{H})$. Under the assumption that λ is sufficiently small so that terms with coefficients λ^2 and greater can be ignored, it follows that $\mathbf{L}^s = \lambda s \gamma^{2(s-1)} \mathbf{G} \mathbf{M}^{s-1} + \gamma^{2s} \mathbf{M}^s$. Letting $\mathbf{L}(\mathbf{G} + ii')^s \equiv [(l_{pq} + ii')^{[s]}]$, $\forall s \in \mathbb{Z}_+$,we have:

$$\Phi_i(\mathbf{G} + ii', \mathbf{H}) - \Phi_i(\mathbf{G}, \mathbf{H}) = \left[\sum_{j=1}^{N} \widehat{m}_{ij} \left(\sum_{s=0}^{\infty} \sum_{q=1}^{N} \left\{ \left(l_{jq} + ii'\right)^{[s]} - l_{jq}^{[s]} \right\} \boldsymbol{\alpha}_{\mathbf{H}q} \right) \right]$$

Since $\mathbf{M}$ is not affected by link formation in $\mathbf{G}$, we can see that:

$$\mathbf{L}(\mathbf{G} + ii')^s - \mathbf{L}^s = \lambda k \gamma^{2(s-1)} \left[\left(\mathbf{G} + ii'\right) - \mathbf{G}\right] \mathbf{M}^{s-1}$$

All entries in the matrix $[(\mathbf{G}+ii')-\mathbf{G}]$ are zero except for 1's in the (i,i') and $(i'i)$ position. It therefore follows that:

$$\Phi_i(\mathbf{G}+ii',\mathbf{H})-\Phi_i(\mathbf{G},\mathbf{H})=\lambda\sum_{s=1}^{\infty}s\gamma^{2(s-1)}\left\{\widehat{m}_{ii}\sum_{q=1}^{N}m_{i'q}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}+\widehat{m}_{ii'}\sum_{q=1}^{N}m_{iq}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}\right\} \tag{32}$$

Also note that player i's Nash equilibrium levels of actions $x_i^*(\mathbf{G},\mathbf{H})$ and $y_i^*(\mathbf{G},\mathbf{H})$ can be written out as:

$$\begin{aligned} x_i^*(\mathbf{G},\mathbf{H}) &= \lambda\sum_{s=1}^{\infty}\left[s\gamma^{2(s-1)}\sum_{q=1}^{N}\sum_{l=1}^{N}g_{il}m_{lq}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}+\gamma^{2s}\sum_{q=1}^{N}m_{iq}^{[s]}\boldsymbol{\alpha}_{\mathbf{H}q}\right] &(33)\\ y_i^*(\mathbf{G},\mathbf{H}) &= b_i(\mathbf{H},\psi,\mathbf{1})+\gamma\sum_{q=1}^{N}m_{iq}x_q^*(\mathbf{G},\mathbf{H}) &(34)\end{aligned}$$

Further, it can be seen that:

$$\begin{aligned} x_i^*(\mathbf{G}+ii',\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) &= \lambda\sum_{s=1}^{\infty}\sum_{q=1}^{N}s\gamma^{2(s-1)}m_{i'q}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q} &(35)\\ x_j^*(\mathbf{G}+ii',\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H}) &= 0,\quad j\neq i,i' &(36)\end{aligned}$$

$$\begin{aligned} y_j^*(\mathbf{G}+ii',\mathbf{H})-y_j^*(\mathbf{G},\mathbf{H}) = \gamma m_{ii}&\left[x_i^*(\mathbf{G}+ii',\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H})\right]\\ &+\gamma m_{ii'}\left[x_{i'}^*(\mathbf{G}+ii',\mathbf{H})-x_{i'}^*(\mathbf{G},\mathbf{H})\right],\quad j\in\mathcal{N}\end{aligned}$$

The following claims follow from these calculations:
Claim 1: The Nash action levels of a player are strictly increasing in own links.
Claim 2: The increment in Nash action of a player from a link depends only on the identity of the partner.

$$x_j^*(\mathbf{G}+ij,\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H})=x_k^*(\mathbf{G}+ik,\mathbf{H})-x_k^*(\mathbf{G},\mathbf{H}) \tag{37}$$

Similarly for y_i^*.
Claim 3: The increment in player i's Nash actions is greater when linking to a partner who is more central in $\mathbf{H}$. A more central player in $\mathbf{H}$ gains more from a link than a less central partner. Formally, if $b_j(\mathbf{H},\psi,\mathbf{1})\geq b_k(\mathbf{H},\psi,\mathbf{1})$, then:

$$\begin{aligned} x_i^*(\mathbf{G}+ij,\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) &\geq x_i^*(\mathbf{G}+ik,\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) &(38)\\ x_j^*(\mathbf{G}+jk,\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H}) &\geq x_k^*(\mathbf{G}+jk,\mathbf{H})-x_k^*(\mathbf{G},\mathbf{H}) &(39)\end{aligned}$$

Similarly for y_i^*.

Claim 4: Player i's Nash actions are greater than that of player j when i is more central than j in H and the neighborhood of j is contained in the augmented neighborhood of i. If $b_i(\mathbf{H},\psi,\mathbf{1}) \geq b_j(\mathbf{H},\psi,\mathbf{1})$, and $\mathbf{N}_j(\mathbf{G}) \subseteq \overline{\mathbf{N}}_i(\mathbf{G})$, then:

$$(x_i^*(\mathbf{G},\mathbf{H}), y_i^*(\mathbf{G},\mathbf{H})) \geq \left(x_j^*(\mathbf{G},\mathbf{H}), y_j^*(\mathbf{G},\mathbf{H})\right) \quad (40)$$

The inequalities are strict if $b_i(\mathbf{H},\psi,\mathbf{1}) > b_j(\mathbf{H},\psi,\mathbf{1})$ and/or $\mathbf{N}_j(\mathbf{G}) \subset \overline{\mathbf{N}}_i(\mathbf{G})$.
Claim 5: If $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$ and $\mathbf{N}_k(\mathbf{G}) \subseteq \overline{\mathbf{N}}_j(\mathbf{G})$, then:

$$x_j^*(\mathbf{G}+ij,\mathbf{H}) \geq x_k^*(\mathbf{G}+ik,\mathbf{H}) \quad (41)$$

The inequality is strict if $b_j(\mathbf{H},\psi,\mathbf{1}) > b_k(\mathbf{H},\psi,\mathbf{1})$ and/or $\mathbf{N}_k(\mathbf{G}) \subset \mathbf{N}_j(\mathbf{G})$.
Claim 6: If $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$ and $\mathbf{N}_k(\mathbf{G}) \subseteq \overline{\mathbf{N}}_j(\mathbf{G})$, then $\Delta(ij) \geq \Delta(ik)$. The inequality is strict if $b_j(\mathbf{H},\psi,\mathbf{1}) > b_k(\mathbf{H},\psi,\mathbf{1})$ and/or $\mathbf{N}_k(\mathbf{G}) \subset \overline{\mathbf{N}}_j(\mathbf{G})$.
Claim 7: If $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$, then:

$$\begin{aligned} \Phi_i(\mathbf{G}+ij,\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H}) &\geq \Phi_i(\mathbf{G}+ik,\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H}) \quad (42) \\ \Phi_j(\mathbf{G}+jk,\mathbf{H}) - \Phi_j(\mathbf{G},\mathbf{H}) &\geq \Phi_k(\mathbf{G}+jk,\mathbf{H}) - \Phi_k(\mathbf{G},\mathbf{H}) \quad (43) \end{aligned}$$

The inequality is strict if $b_j(\mathbf{H},\psi,\mathbf{1}) > b_k(\mathbf{H},\psi,\mathbf{1})$.

Proof of Lemma 1: The inequalities follow by comparing corresponding terms in incremental utilities using Claims 1-7. ■

Proof of Lemma 2: Part (a) follows from (33), (34), and (17). Proof of part (b) can be obtained similarly. ■

Proof of Corollary 1: By hypothesis, $U_k(\mathbf{G}+ik,\mathbf{H}) - U_k(\mathbf{G},\mathbf{H}) > c$ and $U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > c$. It follows from respectively Lemma 2 and (19) that:

$$U_j(\mathbf{G}'+ij,\mathbf{H}) - U_j(\mathbf{G}',\mathbf{H}) > U_j(\mathbf{G}+ij,\mathbf{H}) - U_j(\mathbf{G},\mathbf{H}) \geq U_k(\mathbf{G}+ik,\mathbf{H}) - U_k(\mathbf{G},\mathbf{H}) > c$$

Player i will reciprocate since Lemma 2 and (20) implies that:

$$U_i(\mathbf{G}'+ij,\mathbf{H}) - U_i(\mathbf{G}',\mathbf{H}) > U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) \geq U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > c$$

This proves the result. ■

Proof of Proposition 2(a): Suppose $P_m(\mathbf{H}) = l > 1$ so that $P_m(\mathbf{H}) = \{N-l+1, N-l+2, ..., N-1, N\}$. We have proved the result for $r = 1$ in the main text. Therefore, suppose the result holds for round $r \in \{2, 3, ..., l-1\}$. After r rounds, the interim network is $\mathbf{G}(rN)$ and $\overline{\mathbf{N}}_j(\mathbf{G}(rN)) = \{1, 2, ..., N\}$ for $j \in \{N-r+1, .., N-1, N\}$ and $\mathbf{N}_j(\mathbf{G}(rN)) = \{N-r+1, .., N-1, N\}$ for $j \in \{1, 2, ..., N-r\}$. Therefore players in the set $\{N-r+1, .., N-1, N\}$ are connected in $\mathbf{G}$ to all the remaining players at the end of r

rounds. We will now prove the result for round $r+1$. This round starts with player 1 as the active player. Player 1 can propose a link with any player in the set $\{2, ..., N-r\}$. We claim that player 1 has the most profitable link with player $N-r$ in this set. Since player $N-r \in P_m(\mathbf{H})$, $b_{N-r}(\mathbf{H}, \psi, \mathbf{1}) \geq b_j(\mathbf{H}, \psi, \mathbf{1})$ for all j; further, $\mathbf{N}_j(\mathbf{G}(rN)) = \{N-r+1, .., N-1, N\} \subset \overline{\mathbf{N}}_{N-r}(\mathbf{G}(rN)) = \{N-r, N-r+1, .., N-1, N\}$ for $j \in \{2, ..., N-r-1\}$. Therefore, from (18), for all $j \in \{2, ..., N-r-1\}$:

$$U_1(\mathbf{G}(rN) + 1(N-r), \mathbf{H}) - U_1(\mathbf{G}(rN), \mathbf{H}) \geq U_1(\mathbf{G}(rN) + 1j, \mathbf{H}) - U_1(\mathbf{G}(rN), \mathbf{H})$$

Next we claim that player 1 and player $N-r$ have an incentive to form a link. This will follow from Corollary 1 as follows. By the induction hypothesis, player 1 and player $N-r+1$ have an incentive to form a link in network $\mathbf{G}((r-1)N)$. Note that $b_{N-r}(\mathbf{H}, \psi, \mathbf{1}) = b_{N-r+1}(\mathbf{H}, \psi, \mathbf{1})$, $\mathbf{N}_{N-r+1}(\mathbf{G}((r-1)N)) = \{N-r+2, .., N-1, N\} \subset \overline{\mathbf{N}}_{N-r}(\mathbf{G}((r-1)N)) = \{N-r, N-r+2, .., N-1, N\}$, and $\mathbf{G}((r-1)N) \subset \mathbf{G}(rN)$. Therefore, player 1 and player $N-r$ have a mutually profitable link in $\mathbf{G}(rN)$. To conclude the proof, we show that each subsequent active player $j \in \{2, ..., N-r-1\}$ in round $r+1$ will also form a link with player $N-r$. Suppose this is true for some $k \in \{2, 3, ..., N-r-2\}$. We will show that it is also true for player $k+1$. As before, $b_{N-r}(\mathbf{H}, \psi, \mathbf{1}) \geq b_j(\mathbf{H}, \psi, \mathbf{1})$ for all j. Further, $\mathbf{N}_j(\mathbf{G}(rN+k)) = \{N-r, .., N-1, N\} = \overline{\mathbf{N}}_{N-r}(\mathbf{G}(rN+k))$ for $j \in \{1, 2, ..., k\}$ and $\mathbf{N}_j(\mathbf{G}(rN+k)) = \{N-r+1, .., N-1, N\} \subset \overline{\mathbf{N}}_{N-r}(\mathbf{G}(rN+k))$ for $j \in \{k+2, ..., N-r-1\}$. It follows from (18) player $k+1$ has the most profitable link with player $N-r$ in the set $\{1, 2, ..., k, k+2, ..., N-r-1\}$. Finally, note that player k and $N-r$ had a mutually profitable link in network $\mathbf{G}(rN+k-1)$. Since $b_{k+1}(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$, $\mathbf{N}_k(\mathbf{G}(rN+k-1)) = \{N-r+1, .., N-1, N\} \subset \overline{\mathbf{N}}_{k+1}(\mathbf{G}(rN+k-1)) = \{k+1, N-r+1, .., N-1, N\}$, and $\mathbf{G}(rN+k-1) \subset \mathbf{G}(rN+k)$, it follows from Corollary 1 that player $k+1$ and $N-r$ have a mutually profitable link in network $\mathbf{G}(rN+k)$. ■

Proof of Corollary 2: Let $\mathbf{G}(rN)$ denote the interim network at the end of round r (recall that each round has all N players moving in the order of their index). We have already established that $\mathbf{N}_k(\mathbf{G}(1)) \subseteq \overline{\mathbf{N}}_j(\mathbf{G}(1))$ if $b_j(\mathbf{H}, \psi, \mathbf{1}) > b_k(\mathbf{H}, \psi, \mathbf{1})$. Suppose this property is true for $r = r'$. We will show that it holds for $r = r'+1$. Let $\mathbf{G}(r'N+k-1) \supseteq \mathbf{G}(r'N)$ be the interim network succeeding $\mathbf{G}(r'N)$ when player k is active in round $r'+1$. Suppose k has a profitable link with player i in $\mathbf{G}(r'N+k-1)$. Since j is more central in $\mathbf{H}$, j will move after k in round $r'+1$ in some interim network $\mathbf{G}(r'N+j-1) \supseteq \mathbf{G}(r'N+k-1)$. From Corollary 1, j also has a mutually profitable link with i in $\mathbf{G}(r'N+j-1)$. In addition, in a choice between k and j, any active player in round $r'+1$ realizes greater incremental utility from proposing a link with j due to (18). Therefore $\mathbf{N}_k(\mathbf{G}((r'+1)N)) \subseteq \overline{\mathbf{N}}_j(\mathbf{G}(r'+1)N))$. Therefore the nestedness of neighborhoods requirement in Corollary 1 automatically holds along an improving path and the result follows. ■

Proof of Proposition 2(b): We have already shown that $P_1(\mathbf{H}) \subseteq P_1(\mathbf{G})$. It only remains to show that for each $i \in P_1(\mathbf{G})$ we have $ij \in \mathbf{G}$ if $j \in P_n(\mathbf{G})$ and $ij \notin \mathbf{G}$ otherwise, i.e. players in $P_1(\mathbf{G})$ are connected only to those in $P_n(\mathbf{G})$. There are two cases here: (i) Suppose $P_1(\mathbf{H}) = P_1(\mathbf{G})$. If player i has a mutually profitable link with $j \notin P_n(\mathbf{G})$ in a pws-equilibrium network, then this link was established in some interim network $\mathbf{G}'$ when i was the active player. Since $b_k(\mathbf{H}, \psi, \mathbf{1}) \geq b_i(\mathbf{H}, \psi, \mathbf{1})$ $\forall k \in \mathcal{N} \backslash \{i, j\}$, it follows from Corollary 2 that link jk is mutually profitable in $\mathbf{G}'' \supseteq \mathbf{G}'$. But then player j must be maximally connected in $\mathbf{G}$ which contradicts $j \notin P_n(\mathbf{G})$. (ii) Now suppose $P_1(\mathbf{H}) \subset P_1(\mathbf{G})$. Thus $P_1(\mathbf{G})$ includes players from at least one more partition set in $\mathcal{P}(\mathbf{H})$, say $P_2(\mathbf{H})$. Now suppose $ij \in \mathbf{G}$ for some $i \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ and $j \notin P_n(\mathbf{G})$. If ij was established in interim network $\mathbf{G}'$, then from Corollary 2, j has a mutually profitable link with each $i' \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ in interim networks $\mathbf{G}'' \supseteq \mathbf{G}'$. In case (i) we have already ruled out a link between $j \notin P_n(\mathbf{G})$ and players in $P_1(\mathbf{H})$. This implies $d_i(\mathbf{G}) > d_k(\mathbf{G})$ for $i \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ and $k \in P_1(\mathbf{H})$. But this contradicts $i, k \in P_1(\mathbf{G})$, i.e. the two players should have the same degree in $\mathbf{G}$. ■

Proof of Proposition 3: Consider $l = 1$. Suppose that $P_1(\mathbf{H}) \subset P_1(\mathbf{G})$. We will break the argument into the following steps. (i) We show that $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$, i.e. $P_1(\mathbf{G})$ is constructed from successive elements of the partition $\mathcal{P}(\mathbf{H})$. We can prove this by contradiction. Suppose not and assume that $P_1(\mathbf{G})$ "jumps over" elements of $\mathcal{P}(\mathbf{H})$ such that $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) = \emptyset$ but $P_1(\mathbf{G}) \cap P_q(\mathbf{H}) \neq \emptyset$ for some $q > 2$. If $i \in P_1(\mathbf{G}) \cap P_q(\mathbf{H})$ for $q > 2$, and $j \in P_2(\mathbf{G})$, then $b_i(\mathbf{H}, \psi, \mathbf{1}) > b_j(\mathbf{H}, \psi, \mathbf{1})$ implies that every player k who has a mutually profitable link with j will also have a mutually profitable link with i from Corollary 2. This implies $d_i(\mathbf{G}) \geq d_j(\mathbf{G})$ contradicting that $i \in P_1(\mathbf{G})$. (ii) Next we establish the non-partial overlap property that if $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$, then $P_2(\mathbf{H}) \subset P_1(\mathbf{G})$. Suppose that $P_2(\mathbf{H})$ contains a player $r \notin P_1(\mathbf{G})$. Then r is connected to some non-maximally linked player $r' \notin P_n(\mathbf{G})$, since it is players in $P_1(\mathbf{G})$ who are limited to links with the highest degree players in $\mathbf{G}$. But from Corollary 2, player r' will also have a mutually profitable link with $j \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$. In fact every player who has a link with r also has a mutually profitable link with j. But then $d_j(\mathbf{G}) \geq d_r(\mathbf{G})$ contradicting that $j \in P_1(\mathbf{G})$. It therefore follows that $P_1(\mathbf{G}) = P_1(\mathbf{H}) \cup P_2(\mathbf{H})$ if $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$. (iii) An identical argument establishes the result if $P_1(\mathbf{G}) \cap P_r(\mathbf{H}) \neq \emptyset$ for $r > 2$. An identical argument applies inductively to $l \in \{2, 3, ..., n-1\}$. ■

Proof that G is a NSG with at most one non-singleton component: To prove the implication, recall that an improving path leads to $\mathbf{G}^c$ if $0 \leq c < U_1(\mathbf{G}^e + 12, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H})$ and remains at $\mathbf{G}^e$ if $U_{N-1}(\mathbf{G}^e + (N-1, N), \mathbf{H}) - U_{N-1}(\mathbf{G}^e, \mathbf{H}) \leq c$. Consider an intermediate range of costs. The NSG property for $P_1(\mathbf{G})$ and $P_n(\mathbf{G})$ follows from Proposition 2. Consider the intermediate elements of $\mathcal{P}(\mathbf{G})$ and players $l_1, l_2, .., l_{n-1}$ representing $P_1(\mathbf{G}), P_2(\mathbf{G}), ..., P_{n-1}(\mathbf{G})$ respectively. Consider $l_2 \in P_2(\mathbf{G})$. From the non-partial overlap property, $b_{l_1}(\mathbf{H}, \psi, \mathbf{1}) < b_{l_2}(\mathbf{H}, \psi, \mathbf{1})$. Since each player in $P_1(\mathbf{G})$ is directly linked to all players in $P_n(\mathbf{G})$, there must exist a player $k \notin P_n(\mathbf{G})$ such that $kl_2 \in \mathbf{G}$.We now show that $kl_2 \in \mathbf{G}$ for all $k \in P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$. Suppose not and let $kl_2 \in \mathbf{G}$ but $k \notin P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$. Each $l \in P_2(\mathbf{G}) \cup \cdots \cup P_n(\mathbf{G})$ will satisfy $b_l(\mathbf{H}, \psi, \mathbf{1}) \geq b_{l_2}(\mathbf{H}, \psi, \mathbf{1})$. Therefore, from Corollary 2, player k will have mutually profitable links with all players implying that $k \in P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$, a contradiction. Continuing inductively in this manner generates a NSG with the neighborhoods of low degree players nested in the neighborhoods of high degree players. Finally, note that there can be at most one non-singleton component. If to the contrary there exists at least two non-singleton components in a pws-equilibrium graph, then we can identify players i, j, k, l such that $ij \in \mathbf{G}$, $kl \in \mathbf{G}$, and $b_i(\mathbf{H}, \psi, \mathbf{1}) \leq b_k(\mathbf{H}, \psi, \mathbf{1}) \leq b_j(\mathbf{H}, \psi, \mathbf{1}) \leq b_l(\mathbf{H}, \psi, \mathbf{1})$. But then players j and l can mutually benefit from a link. ■

Proof of Proposition 4: Let $k_1, k_2, ..., k_q \in \mathcal{N}_i(\mathbf{G}) \setminus \mathcal{N}_j(\mathbf{G})$ and $\widetilde{\mathbf{G}} = \mathbf{G} - \sum_{s=1}^{q} ik_s$. From (17), $U_i(\widetilde{\mathbf{G}}, \mathbf{H}) - d_j(\mathbf{G})c > U_j(\widetilde{\mathbf{G}}, \mathbf{H}) - d_j(\mathbf{G})c$. It follows that:

$$\begin{aligned}
U_i(\mathbf{G},\mathbf{H}) - d_i(\mathbf{G})c &= U_i(\mathbf{G},\mathbf{H}) - U_i(\widetilde{\mathbf{G}},\mathbf{H}) - [d_i(\mathbf{G}) - d_j(\mathbf{G})]\, c + U_i(\widetilde{\mathbf{G}},\mathbf{H}) - d_j(\mathbf{G})c \\
&= \left[U_i\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q} ik_s, \mathbf{H}\right) - U_i\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-1} ik_s, \mathbf{H}\right) - c\right] \\
&\quad + \left[U_i\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-1} ik_s, \mathbf{H}\right) - U_i\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-2} ik_s, \mathbf{H}\right) - c\right] \\
&\quad + \cdots + \left[U_i\left(\widetilde{\mathbf{G}} + ik_1, \mathbf{H}\right) - U_i\left(\widetilde{\mathbf{G}}, \mathbf{H}\right) - c\right] + U_i(\widetilde{\mathbf{G}},\mathbf{H}) - d_j(\mathbf{G})c \\
&> [d_i(\mathbf{G}) - d_j(\mathbf{G})]\, \underline{u} + U_j(\widetilde{\mathbf{G}},\mathbf{H}) - d_j(\mathbf{G})c \\
&\geq [d_i(\mathbf{G}) - d_j(\mathbf{G})]\, \overline{u} + U_j(\widetilde{\mathbf{G}},\mathbf{H}) - d_j(\mathbf{G})c \\
&\geq \left[U_j\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q} ik_s, \mathbf{H}\right) - U_j\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-1} ik_s, \mathbf{H}\right)\right] \\
&\quad + \left[U_j\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-1} ik_s, \mathbf{H}\right) - U_j\left(\widetilde{\mathbf{G}} + \sum_{s=1}^{q-2} ik_s, \mathbf{H}\right)\right] \\
&\quad + \left[U_j\left(\widetilde{\mathbf{G}} + ik_1, \mathbf{H}\right) - U_j\left(\widetilde{\mathbf{G}}, \mathbf{H}\right)\right] + U_j(\widetilde{\mathbf{G}},\mathbf{H}) - d_j(\mathbf{G})c \\
&= U_j(\mathbf{G},\mathbf{H}) - d_j(\mathbf{G})c
\end{aligned}$$

Therefore, a player in a higher echelon of the degree partition of the equilibrium NSG network $\mathbf{G}$ gets a strictly greater payoff than a player in a lower echelon. ■

Proof of Proposition 6: (a) Suppose $\mathcal{D}\left(\mathbf{G}_{(0)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(1)}\right)$. We will prove by contradiction. Let $t' \in \{1, 2, ...\}$ be the first time index such that $\mathcal{D}\left(\mathbf{G}_{(t')}\right) \not\succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(t'+1)}\right)$. Both $\mathbf{G}_{(t')}$ and $\mathbf{G}_{(t'+1)}$ start from $\mathbf{G}^e$. Let i be the *first* family in generation $t'+1$ that forms a mutually profitable link with family j that did not exist in generation t'. Suppose this happens after r rounds. Therefore, the interim networks on the paths leading respectively to $\mathbf{G}_{(t')}$ and $\mathbf{G}_{(t'+1)}$ are the same until interim network $\mathbf{G}(rN + i - 1)$ is reached and diverges after that when $i(t'+1)$ links with $j(t'+1)$. Also note that $\mathbf{H}_{(t'+1)} = \mathbf{G}_{(t')}$. Therefore:

$$U_{i(t'+1)}\left(\mathbf{G}(rN+i-1) + i(t'+1)j(t'+1), \mathbf{G}_{(t')}\right) - U_{i(t'+1)}\left(\mathbf{G}(rN+i-1), \mathbf{G}_{(t')}\right) > c$$

and similarly for $j(t'+1)$. By the definition of t', $\mathcal{D}\left(\mathbf{G}_{(t'-1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(t')}\right)$, i.e. $\mathbf{G}_{(t'-1)} = \mathbf{H}_{(t')} \subset \mathbf{H}_{(t'+1)} = \mathbf{G}_{(t')}$. It then follows from Lemma 2(b) that:

$$\begin{aligned}
&U_{i(t')}\left(\mathbf{G}(rN+i-1) + i(t')j(t'), \mathbf{G}_{(t'-1)}\right) - U_{i(t')}\left(\mathbf{G}(rN+i-1), \mathbf{G}_{(t'-1)}\right) \\
&\qquad > U_{i(t'+1)}\left(\mathbf{G}(rN+i-1) + i(t'+1)j(t'+1), \mathbf{G}_{(t')}\right) - U_{i(t'+1)}\left(\mathbf{G}(rN+i-1), \mathbf{G}_{(t')}\right) > c
\end{aligned}$$

and similarly for $j(t')$. Thus $i(t')j(t') \in \mathbf{G}_{(t')}$, a contradiction. (b) In this case, $\mathbf{G}_{(0)} = \mathbf{H}_{(1)} \subset \mathbf{H}_{(2)} = \mathbf{G}_{(1)}$. From Lemma 2(b), each link in $\mathbf{G}_{(1)}$ that was formed by a pair of players in generation 1 will also be profitable for generation 2 and thus exist in $\mathbf{G}_{(2)}$ yielding $\mathcal{D}\left(\mathbf{G}_{(2)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(1)}\right)$. The rest follows by induction. ■

Proof of Proposition 7: Note that $\mathbf{H}^{\mathbf{s}} \subseteq \mathbf{H}_{(1)}$, since the star network is first order dominated by all connected NSG networks. Therefore, given $c \leq c_s$, the least and most connected players in $\mathbf{H}_{(1)}$ have a mutually profitable link in $\mathbf{G}_{(1)}$. It follows that $\mathbf{G}_{(1)}$ is non-empty with no isolated players. Therefore, from Theorem 1, $\mathbf{G}_{(1)}$ has a connected NSG architecture. Continuing inductively, $\mathbf{G}_{(t)}$ is connected for all t. (a) Note that $\mathcal{D}\left(\mathbf{G}_{(0)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}^{e}\right)$. Consider generation 1 and any stage $s+1$ of the network formation game when the current network is $\mathbf{G}(s)$, the active player is $i(1)$, and the reactive player to whom a link is proposed is $j(1)$. Then respectively from Lemma 2(a) and (b), and the definition of $\underline{c}$:

$$\begin{aligned} U_{i(1)}\left(\mathbf{G}(s)+i(1)j(1), \mathbf{G}_{(0)}\right)-U_{i(1)}\left(\mathbf{G}(s), \mathbf{G}_{(0)}\right) &> U_{i(1)}\left(\mathbf{G}^{e}+i(1)j(1), \mathbf{G}_{(0)}\right)-U_{i(1)}\left(\mathbf{G}^{e}, \mathbf{G}_{(0)}\right) \\ &> U_{i(1)}\left(\mathbf{G}^{e}+i(1)j(1), \mathbf{G}^{e}\right)-U_{i(1)}\left(\mathbf{G}^{e}, \mathbf{G}^{e}\right)=\underline{c} \geq c \end{aligned}$$

Similarly for player $j(1)$. Therefore any pair of unlinked players have an incentive to connect and $\mathbf{G}_{(1)}$ is complete. It follows that $\mathcal{D}\left(\mathbf{G}_{(1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(0)}\right)$ and thus $\mathcal{D}\left(\mathbf{G}_{(t+1)}\right) \succeq_{FOD} \mathcal{D}\left(\mathbf{G}_{(t)}\right)$ from Proposition 6 for all t. Since $\mathbf{G}_{(1)}=\mathbf{G}^{c}$, it follows that $\mathbf{G}_{(t)}=\mathbf{G}^{c}$ for all t. (b) The argument is similar. ■

Proof of Proposition 8: Recall that since γ is sufficiently small, all terms of order γ^2 and above are ignored. Since $\mathbf{N}_j(\mathbf{G}) \subseteq \mathbf{N}_k(\mathbf{G})$, for distinct players i, i', j, and k:

$$\begin{aligned} n_{ii}(\mathbf{G}+ij, \mathbf{H})-n_{ii}(\mathbf{G}, \mathbf{H}) &\leq n_{ii}(\mathbf{G}+ik, \mathbf{H})-n_{ii}(\mathbf{G}, \mathbf{H}) \\ n_{ij}(\mathbf{G}+ij, \mathbf{H})-n_{ij}(\mathbf{G}, \mathbf{H}) &\leq n_{ik}(\mathbf{G}+ik, \mathbf{H})-n_{ik}(\mathbf{G}, \mathbf{H}) \\ n_{jj}(\mathbf{G}+ij, \mathbf{H})-n_{jj}(\mathbf{G}, \mathbf{H}) &\leq n_{kk}(\mathbf{G}+ik, \mathbf{H})-n_{kk}(\mathbf{G}, \mathbf{H}) \\ n_{ii'}(\mathbf{G}+ij, \mathbf{H})-n_{ii'}(\mathbf{G}, \mathbf{H}) &\leq n_{ii'}(\mathbf{G}+ik, \mathbf{H})-n_{ii'}(\mathbf{G}, \mathbf{H}) \end{aligned}$$

Since $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$, it follows that $\alpha_{H_j} \leq \alpha_{H_k}$. The result follows from inspection of (26) and (27). ■

Proof of Proposition 9: For any $ij \notin \mathbf{G}$, then $x_i^*(\mathbf{G}+ij, \mathbf{H}) < x_i^*(\mathbf{G}, \mathbf{H})$ and $x_k^*(\mathbf{G}+ij, \mathbf{H}) = x_k^*(\mathbf{G}, \mathbf{H})$ for $k \neq i, j$. Further, for all $j \in \mathcal{N}$, $y_j^*(\mathbf{G}+ij, \mathbf{H}) < y_j^*(\mathbf{G}, \mathbf{H})$. Note that:

$$\begin{aligned} U_i(\mathbf{G}+ij, \mathbf{H})-U_i(\mathbf{G}, \mathbf{H}) = {} & \frac{1}{2}\left[(x_i(\mathbf{G}+ij, \mathbf{H}))^2-(x_i(\mathbf{G}, \mathbf{H}))^2\right]+\frac{1}{2}\left[(y_i(\mathbf{G}+ij, \mathbf{H}))^2-(y_i(\mathbf{G}, \mathbf{H}))^2\right] \\ & +\gamma\left[x_i(\mathbf{G}+ij, \mathbf{H}) y_i(\mathbf{G}+ij, \mathbf{H})-x_i(\mathbf{G}, \mathbf{H}) y_i(\mathbf{G}, \mathbf{H})\right] \end{aligned}$$

Note that all three terms have negative values. Therefore, $U_i(\mathbf{G}+ii', \mathbf{H}) < U_i(\mathbf{G}, \mathbf{H})$ and there is no incentive to form any links in $\mathbf{G}$. ■

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