The Value of Price Discrimination in Large Social Networks

Jiali Huang
Department of Industrial and Systems Engineering
University of Minnesota, Minneapolis, MN, 55455, huan1106@umn.edu

Ankur Mani
Department of Industrial and Systems Engineering
University of Minnesota, Minneapolis, MN, 55455, amani@umn.edu

Zizhuo Wang
School of Data Science
The Chinese University of Hong Kong, Shenzhen, China, 518172, wangzizhuo@cuhk.edu.cn

We study the value of price discrimination in large social networks. Recent trends in industry suggest that increasingly firms are using information about social network to offer personalized prices to individuals based upon their positions in the social network. In the presence of positive network externalities, firms aim to increase their profits by offering discounts to influential individuals that can stimulate consumption by other individuals at a higher price. However, the lack of transparency in discriminative pricing may reduce consumer satisfaction and create mistrust. Recent research has focused on the computation of optimal prices in deterministic networks under positive externalities. We would like to answer the question: how valuable is such discriminative pricing? We find, surprisingly, that the value of such pricing policies (increase in profits due to price discrimination) in very large random networks are often not significant. Particularly, for Erdös-Renyi random networks, we provide the exact rates at which this value decays in the size of the networks for different ranges of network densities. Our results show that there is a non-negligible value of price discrimination for a small class of moderate-sized Erdös-Renyi random networks. We also present a framework to obtain bounds on the value of price discrimination for random networks with general degree distributions and apply the framework to obtain bounds on the value of price discrimination in power-law networks. Our numerical experiments demonstrate our results and suggest that our results are robust to changes in the model of network externalities.

Key words: Personalized pricing in networks; Value of price discrimination; Social networks; Centrality

1. Introduction

The use of social network information in operations and marketing has become increasingly prevalent in recent years. The increasing popularity of social platforms, such as Facebook, Twitter, Wechat and Instagram, has enabled convenient online social interactions among people. Those social interactions make individuals more likely to influence or to be influenced by peers in the

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social network. In the meantime, the rapid development of information technology has enabled firms to utilize social network information of their consumers in their marketing and operations decisions. Indeed, it is reported that firms such as Gilt Groupe and PALMS resort have been using such information for targeted advertising, steering, identifying influencers in the social network for product promotion and viral marketing.[1] Other Internet companies also find using social network information a potential way to improve their operations and marketing efficiencies.[2] Fainmesser and Galeotti (2020) provide a summary of industrial practice of influencer marketing, including the use of discriminative pricing strategies.

Among the different uses of social network information in operations and marketing, one important use is in pricing. In many situations, firms would like to sell products to consumers on the social platform (either their own products or serve as a platform to sell products of a third party). In such cases, firms may use the information about social network to inform their pricing decisions. For example, firms would like to know which individual should be targeted for a particular promotion or which individual might be willing to pay a higher price. Apparently, such information is very helpful for firms to wisely use their resources (e.g., campaign budget or limited webpage space) to the maximal benefit.

However, despite the potential benefit, the use of social network information in pricing has raised serious concerns. For government agencies, such practices inherently cause inequity and may violate certain regulations (see, e.g., Obama White House Report 2015). Consumers are concerned about their privacy as well as the prospect that they might be offered a higher price than their peers.[3] Given such concerns, firms face the dilemma of whether to use social network information in their pricing decisions. To solve the dilemma, it is important for the firm to understand how much potential gain there is from the use of social network information in pricing — if the gain is small, then there may not be much motivation for such practices in the first place.

The paper aims to shed light on the above question. Specifically, we investigate how much value can be added if the firm utilizes social network information to inform their pricing decisions (to offer personalized prices based on the positions of the individuals in a social network). In particular, we try to answer the following questions.

[1] See `https://www.wired.com/2012/04/ff-klout/`.

[2] See `https://www.forbes.com/sites/neilhowe/2016/03/31/brands-are-under-the-social-influence/?sh=346832245f49`.

[3] See `https://hbr.org/2018/05/when-customers-are-and-arent-ok-with-personalized-prices` for results from a large-scale experiment.

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1. Given a specific social network, what is the value for the firm to offer personalized pricing to each individual consumer compared to offer a uniform price to all consumers?

2. What is the value of information about the network structure if used for pricing?

3. Which type of network (structure) will generate more additional value if the firm uses discriminative pricing?

To answer these questions, we consider a model that is widely used in operations management to study pricing decisions under network effects. In the model, there is a network of consumers. Each consumer may influence other consumers with different weights. The firm sells a divisible product to all consumers, and it has the ability to offer personalized prices to each individual consumer (e.g., by showing personalized coupons or promotions). Each consumer has a utility function that depends on the purchased amount of the product by him/herself as well as the purchase amount by his/her neighbors in the social network.

Then we consider two pricing strategies for the firm. In the first strategy, the firm knows the social network structure and utilizes this information to offer personalized prices to maximize its profit. In the second strategy, the firm does not apply personalized offering and simply offers a uniform price to all consumers. Note that not offering personalized prices may be due to ignorance of the network structure or simply because the firm decides not to offer personalized prices. Under our model assumptions, the optimal uniform price does not depend on the network information; so if the firm decides not to offer personalized prices, then it can set prices ignoring the network effects. In either case, the consumers decide the purchase amount based on their utility function (with network influences). Then we compare the profits obtained by the firm using these two strategies (namely, optimal personalized prices versus optimal uniform prices). The difference of the two profits will be the value of price discrimination. It can also be viewed as the value of network information for the corresponding network.

The value of price discrimination is always non-negative. Nevertheless, the magnitude of the value depends largely on the network structure. First, we find that for certain networks, there is no value of price discrimination. That is, for those networks, even if the firm knows the network structure and can offer discriminative prices, the optimal action is to offer a uniform price for all consumers. We identify a critical property for this class of networks — the number of walks of any given length starting from each node must be equal to the number of walks of the same length ending at that node. This result gives a precise characterization of when the value of price discrimination is zero for a given network.

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Having understood the value of price discrimination for a given network, we further consider the value of price discrimination for random networks. In particular, we focus on a class of random networks — the Erdös-Renyi networks. In an Erdös-Renyi network, a consumer has a unit influence on another consumer with a certain probability (we call it the *influence probability*). We consider the asymptotic value of price discrimination for large Erdös-Renyi networks (the relative improvement in profit of discriminative pricing over uniform pricing). We have the following findings for Erdös-Renyi networks:

1. When the influence probability is below a certain threshold, the asymptotic value of price discrimination increases with the influence probability.

2. When the influence probability is above a certain threshold, the asymptotic value of price discrimination decreases with the influence probability.

3. We identify the range of the influence probability in which the asymptotic value of price discrimination reaches the maximum.

4. In all cases, the value of price discrimination is asymptotically vanishing as the network size becomes large.

To explain our results, we note that when the network is very sparse (the influence probability is below a threshold), the network is very fragmented and the social influence of any individual is contained within his/her closest neighbors, leading to small gains from price discrimination (we show that the gain of discriminative pricing depends largely on the number of long paths in the network); when the network is very dense (the influence probability is above a threshold), the network becomes very balanced, also leading to small value of price discrimination (we show that the existence of cycles will reduce the gain of discriminative pricing); when the density of network is intermediate, there exists long paths but not as many cycles, and the value of price discrimination reaches its maximum. However, even at the maximum, the value of discriminative pricing still asymptotically decays to zero, indicating that for large Erdös-Renyi networks, with high chance, there is little value of applying discriminative prices. Meanwhile, complementary to the asymptotic results, we find that for Erdös-Renyi networks with a certain range of influence probability, the rate at which the value of price discrimination decays is slow, suggesting that the value of price discrimination may be non-negligible for small- or moderate-sized networks.

We also extend our analysis to random networks with general degree distributions. Specifically, we provide a general framework to obtain upper bounds for the value of price discrimination, based on the maximum degrees and the second moments of the degree distributions of the random networks. As an application of this general framework, we investigate the asymptotic value of

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price discrimination for random networks with power-law degree distributions. We show that for power-law networks, the value of price discrimination asymptotically vanishes as the size of the network increases. More specifically, we provide the rates of the decay for power-law networks with different ranges of parameters. These results suggest that the value of price discrimination may not be significant for a broader class of large random networks.

In addition to our theoretical results, we perform numerical experiments to demonstrate our results. We first use synthetic data and show that the numerical results match our theoretical results. Then we use social networks from real data sets (that are often not Erdös-Renyi networks) to compute the value of price discrimination. We show that for large networks, the value of price discrimination is often quite small. However, for moderate-sized networks, there could be an non-negligible value of price discrimination. To show the robustness of our results, we also numerically test variants of our model and observe similar results.

The remainder of the paper is organized as follows. In Section 2, we review related literature to this work. In Section 3, we introduce the basic model considered in this paper. In Section 4, we study the conditions under which there is no value of price discrimination. Then in Section 5, we consider a class of random networks — the Erdös-Renyi networks and study the asymptotic value of price discrimination. We present the main results in Section 5.1 and provide the key proof concepts in Section 5.2. In Section 6, we provide a general framework to study the value of price discrimination in general random networks and apply the framework to power-law networks. We conducts some numerical experiments in Section 7. We conclude the paper in Section 8.

2. Literature Review

The study of network effects has been an active research topic in recent years. Through cascades of influence, network effects can shape critical outcomes in a social network such as the spread of information, ideas and disease (see, e.g., Pastor-Satorras and Vespignani 2001, Chamley 2004, Banerjee et al. 2013, Muchnik et al. 2013), choice or adoption of products by consumers (see, e.g., Rogers 1976, Bapna and Umyarov 2015, Wang and Wang 2017), and so on.

Our work is related to the literature considering positive network externalities, as introduced in Farrell and Saloner (1985) and Katz and Shapiro (1985). Specifically, our work is related to the work on optimal marketing strategies with positive network externalities. There are often two objectives in such studies, one aims at *influence maximization* across the network, while the other aims at *revenue maximization*. In the following, we discuss the literature on both streams.

Influence maximization problems consider diffusion of influence, and aim at identifying the best *seeds* to maximize the spread of social influence. For example, Domingos and Richardson (2001)

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introduce the concept of influence maximization in virtual marketing by initially targeting the seeds and then triggering the influence cascade among consumers. Such problems have been widely studied subsequently in various settings, see, e.g., Richardson and Domingos (2002) and Banerjee et al. (2013). Particularly, Kempe et al. (2003) show that this problem is computationally complex, and provide provable approximation algorithm for the problem. Recently, Akbarpour et al. (2018) consider the value of network information for diffusion problems and suggest that a random seeding strategy with a few more seeds can prompt a larger cascade than optimally targeting, the result of which is similar in spirit to ours but in a different setting.

Our work mainly belongs to the revenue maximization stream. Revenue maximization problems not only consider the diffusion of influence, but also aim at maximizing the revenue. In this stream, there has been much research on efficient marketing strategies using network effects, especially on the influence, exploit or pricing strategy in the setting of sequential purchase decisions (see, e.g., Hartline et al. 2008, Arthur et al. 2009, Haghpanah et al. 2013, Crapis et al. 2016, Zhou and Chen 2018). Such problems are often computationally complex and most literature focus on the computational approaches for such problems. There are also a significant amount of works on pricing in the presence of network effect and simultaneous purchase decisions (see, e.g., Campbell 2013, Du et al. 2016, Chen et al. 2018, Cohen and Harsha 2020). Again, most such research focus on the computational aspects of the problem.

Our work is closely related to the static pricing problem of selling a divisible product to a group of consumers with positive network externalities, in particular, the works of Candogan et al. (2012), Bloch and Quérou (2013) and Fainmesser and Galeotti (2015). In particular, we build upon the model in Candogan et al. (2012) which is a deterministic model and we introduce structural randomness in the model. These works consider a two-stage game, in which the monopolist first chooses the prices and then the consumers, embedded in a social network, make purchasing decisions simultaneously. Candogan et al. (2012) and Bloch and Quérou (2013) assume full network information, while Fainmesser and Galeotti (2015) assume incomplete network information and consider a configuration network model. Candogan et al. (2012) and Fainmesser and Galeotti (2015) consider the amount of consumption of the consumers and assume quadratic utility in the consumption quantity, while Bloch and Quérou (2013) primarily consider a binary purchase decision and assume a linear utility function. In these works, prices can be set differently based upon individuals' positions in the social network. The question of interest is to characterize and identify optimal prices and profits in the networks and the complexity of computing optimal prices. In contrast, our main goal is to quantify the value of price discrimination, that is, to understand how much potential

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value can be added by discriminative pricing. We notice a few recent papers addressing similar problems, but in different settings and approaches. For example, Momot et al. (2020) compare the value of degree-information and conspicuity-information in the setting of selective selling of exclusive products with negative network externalities. Alizamir et al. (2018) analyze a firm's optimal pricing problem for a new service with local network effect in a non-stationary dynamic setting, and study the impact of network structure on the firm's revenue and pricing decisions. In contrast, our work studies the value of discriminative pricing in random social networks.

Our work is also connected to some literature in marketing and economics, such as network games (see, e.g., Ballester et al. 2006, Sundararajan 2007, Galeotti et al. 2010) and personalized pricing with heterogeneous customer valuations (see, e.g., Choudhary et al. 2005, Elmachtoub et al. 2018). Moreover, in developing the asymptotic value of price discrimination in random networks, we establish connections with graph theory and random graph theory, in particular, the literature on counting the number of walks of different lengths in a network (see, e.g., Fiol and Garriga 2009, Levin and Peres 2017), on spectral graph theory (see, e.g., Krivelevich and Sudakov 2003, Chung et al. 2004, Preciado and Rahimian 2017), and on network centrality (see, e.g., Bonacich 1987). We will illustrate more detailed connections to these literatures when discussing the proofs of our theoretical results.

3. Model

In this section, we introduce our monopolist-consumer model with network externalities. We characterize the consumption equilibrium, optimal discriminative pricing, and optimal uniform pricing under our model. We also define the value of price discrimination based on the increase in the profit under optimal discriminative pricing over that under optimal uniform pricing, namely the regret and the fractional regret.

3.1. Basic Model

We consider the pricing problem of a monopolist who sells a divisible product to consumers in a social network. Our basic model is motivated by Candogan et al. (2012), Bloch and Quérou (2013) and Fainmesser and Galeotti (2015). In our model, there are n consumers in a directed social network with non-empty adjacency matrix $G \in \mathbb{R}^{n\times n}_{\geq 0}$. The element G_{ij} represents the influence of consumer j on consumer i. The monopolist chooses a price p_i for each consumer i, and each consumer i chooses the consumption level x_i. The preferences of the consumers are represented by the following utility function:

$$u_i(x_i, \mathbf{x}_{-i}, p_i) = ax_i - x_i^2 + 4\rho x_i \sum_{j\neq i} \frac{G_{ij}}{\|G+G^T\|} x_j - p_i x_i,$$

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where $\mathbf{x}_{-i}$ is a vector representing the consumption levels of all consumers other than i, $\|G+G^T\|$ is the spectral norm of $(G+G^T)$, $a>0$ is a constant representing the strength of stand-alone utility, and $\rho \in (0,1)$ is a positive network externality coefficient representing the strength of network effect.

Our model is built upon the model in Candogan et al. (2012), except that we assume homogeneous consumer preferences, include the spectral norm of $(G+G^T)$ in the utility function to normalize the network effect (this is necessary as we will later extend the model into a sequence of pricing problems over growing networks), and we introduce ρ as the network externality coefficient. The introduction of ρ is consistent with the model in Bloch and Quérou (2013) and Fainmesser and Galeotti (2015). As for the network information, our model is accordant with Candogan et al. (2012) and Bloch and Quérou (2013) in assuming full network information, i.e., the adjacency matrix G.

We focus on the discriminative pricing strategy based upon network positions. To block other unnecessary idiosyncrasies, we assume that all consumers have identical preferences and their decisions differ only because of their positions in the network, i.e., a and ρ are homogeneous across consumers. With the homogeneous assumption, we can isolate the impact of network structure and avoid confounding effects due to heterogeneity among consumers. The consumers choose their consumption levels that maximize their utilities given the prices offered to them and the consumption levels of their peers in the network. Thus the consumers are in a consumption game that is completely represented by $\{\mathcal{N},[0,\infty)_{i\in\mathcal{N}},\{u_i\}_{i\in\mathcal{N}}\}$, where $\mathcal{N}=\{1,2,...,n\}$ is the set of consumers, $[0,\infty)$ is the set of possible consumption levels for each consumer, and u_i is the utility function for consumer i. The utility function $\{u_i\}_{i\in\mathcal{N}}$ for each consumer i is completely identified given the parameters a, ρ, price vector $\mathbf{p}\in\mathbb{R}^n$ and the network adjacency matrix G. According to Candogan et al. (2012), given a, ρ and G, there is a unique consumption equilibrium given by

$$\mathbf{x}^*(\mathbf{p}) = \frac{1}{2}\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}(a\mathbf{1}-\mathbf{p}). \tag{3.1}$$

This equilibrium is well defined because $\left(I - \frac{2\rho}{\|G+G^T\|}G\right)$ is a positive definite matrix. This is because $\rho<1$, and the spectral norm of $2G$ is at most the spectral norm of $(G+G^T)$ (see Theorem 2 in Fan 1950). The assumption $\rho<1$ is weaker than Assumption 1 in Candogan et al. (2012) which requires that the row sums of the adjacency matrix G be uniformly bounded by twice the quadratic coefficient in the utility function (which is 1 in our model). If $\rho<1$ is violated, then $\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}$ may not be well defined and it would lead to an equilibrium where consumers consume infinite quantity.

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Next, we consider the seller's problem. We assume that the monopolistic seller can observe the network structure and can produce arbitrary quantity at a constant marginal cost $c < a$. Then the monopolist's pricing problem is defined as:

$$\max_{\mathbf{p}} \; (\mathbf{p} - c\mathbf{1})^T \mathbf{x}^*(\mathbf{p}).$$

Again, as shown in Candogan et al. (2012), the optimal price vector is

$$\begin{aligned}
\mathbf{p}^* &= \left(\frac{a+c}{2}\right)\mathbf{1} + \left(\frac{a-c}{8}\right)\frac{4\rho}{\|G+G^T\|} G\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right) \\
&\quad - \left(\frac{a-c}{8}\right)\frac{4\rho}{\|G+G^T\|} G^T\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right) \\
&= \left(\frac{a+c}{2}\right)\mathbf{1} + \left(\frac{a-c}{2}\right)\frac{\rho}{\|G+G^T\|}\left(G-G^T\right)\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right),
\end{aligned} \tag{3.2}$$

where $\mathcal{K}(G,\alpha) = (I-\alpha G)^{-1}\mathbf{1} = \sum_{i=0}^{\infty} (\alpha G)^i \mathbf{1}$ is the Bonacich centrality vector proposed in Bonacich (1987). According to the definition, the Bonacich centrality of consumer i is the discounted sum of weighted walks of all lengths ending at consumer i. The discount factor exponentially decreases in the length of the walk (for a walk of length k, the discount factor is α^k where α is the discount rate) and the weight of a walk is the product of the weights of the edges in the walk. In particular, if the adjacency matrix G is binary, then all walks have weight 1 and the Bonacich centrality of consumer i is the discounted sum of number of walks of all lengths ending at consumer i.

We point out that the optimal price vector (3.2) consists of three parts: a common charge for all consumers (the first term), a markup term proportional to the influence received by the consumers (the second term), and a discount term proportional to the influence a consumer exerts on other consumers (the third term). A consumer influencing more central peers gets a higher discount than a consumer influencing less central peers; and a consumer influenced by more central peers gets a higher mark up than a consumer influenced by less central peers. This structure has also been observed in Candogan et al. (2012), Bloch and Quérou (2013) and Fainmesser and Galeotti (2015). Plugging the optimal price into the objective function, we have that the optimal profit of the monopolist is

$$\pi^* = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{1}^T\left(I - \frac{\rho}{\|G+G^T\|}(G+G^T)\right)^{-1}\mathbf{1} = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=0}^{\infty}\left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T\left(G+G^T\right)^k\mathbf{1}. \tag{3.3}$$

We note that the optimal profit is proportional to the discounted sum of weighted walks of all lengths in the network with the adjacency matrix $(G+G^T)$.

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In practice, the monopolist may not be able to provide discriminative pricing due to various reasons. In such cases, under the assumption that $c < a$, according to Candogan et al. (2012), the optimal uniform price vector is

$$\mathbf{p}_0 = \frac{a+c}{2}\mathbf{1}.$$

We note that the optimal uniform price vector does not depend on the network information.[4] This implies that there is no value of network information if the monopolist cannot use discriminative pricing. However, this does not imply that the monopolist's profit under optimal uniform price vector is independent of the network structure. Actually, under the optimal uniform price vector $\mathbf{p}_0$, the monopolist's profit is

$$\pi_0 = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1} \mathbf{1} = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=0}^{\infty} \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}. \tag{3.4}$$

Thus the monopolist's profit using optimal uniform price is proportional to the discounted sum of weighted walks of all lengths in the directed network with adjacency matrix G. We next introduce the performance metrics used to evaluate pricing policies.

3.2. Regret and Fractional Regret

We define the monopolist's regret for a price vector $\mathbf{p}$ as the difference between the optimal profit and the profit under the price vector $\mathbf{p}$. That is, the regret of the monopolist under price vector $\mathbf{p}$ is given by

$$R(\mathbf{p}) = \pi^* - \pi(\mathbf{p}),$$

where $\pi(\mathbf{p}) = (\mathbf{p} - c\mathbf{1})^T \mathbf{x}^*(\mathbf{p})$ and $\mathbf{x}^*(\mathbf{p})$ is defined in (3.1).

We also consider another performance metric — the fractional regret. The fractional regret of the monopolist under price vector $\mathbf{p}$ is

$$R_F(\mathbf{p}) = 1 - \frac{\pi(\mathbf{p})}{\pi^*}.$$

The value of price discrimination can now be evaluated using the two performance metrics. We will consider the following two metrics to quantify the value of price discrimination:

1. Monopolist's regret under optimal uniform pricing, i.e.,

$$R(\mathbf{p}_0) = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G+G^T\|}\left(G+G^T\right)\right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1} \mathbf{1}\right]. \tag{3.5}$$

[4] Without the assumption that all consumers have identical preferences, the optimal uniform price vector may depend on the network structure G.

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2. Monopolist's fractional regret under optimal uniform pricing, i.e.,

$$R_F(\mathbf{p}_0) = 1 - \frac{\mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|} G\right)^{-1} \mathbf{1}}{\mathbf{1}^T \left(I - \frac{\rho}{\|G+G^T\|} (G+G^T)\right)^{-1} \mathbf{1}}. \tag{3.6}$$

Since the monopolist does not use the network information for optimal uniform pricing, the value of price discrimination is also the value of network information. This equivalence is important because collecting and using network information may be operationally expensive for the monopolist and may create consumer privacy concerns. With the value of price discrimination, one can answer whether the cost of collecting network information is justified.

4. Value of Price Discrimination for Deterministic Networks

In this section, we study how the network structure plays a role in determining the value of price discrimination. In particular, we provide a characterization of networks for which there is no value of price discrimination. This does not imply that the network effect is absent but that price discrimination does not generate additional profit in those networks. In Candogan et al. (2012), the authors show that when the network G is symmetric, the optimal discriminative price vector equals the optimal uniform price vector, suggesting that there is no value of price discrimination if the network is symmetric. Therefore, symmetry is a sufficient condition under which there is no value of price discrimination. However, we find that there is a larger class of networks with no value of price discrimination, with symmetric networks forming a subclass.

In the following, we use the monopolist's regret under uniform pricing $R(\mathbf{p}_0)$ as the measure of value of price discrimination. We make the following definition.

Definition 1 (Price Discrimination Free Network). A network is a price discrimination free network if for any $\rho \in (0,1)$, $a > c > 0$, the value of price discrimination is zero, i.e., $R(\mathbf{p}_0) = 0$.

In the following, we study necessary and sufficient conditions for a network to be price discrimination free. At first sight, the price discrimination free property should have some connection with certain "symmetric" properties of networks such as symmetry, identical centrality for nodes or balanced in-degree and out-degrees for nodes. However, as the following five examples show, none of these properties is equivalent to the price discrimination free property. In all the examples below, we assume that all edges have weight 1, and that $a = 5$, $c = 4$ and $\rho = 0.9$.

Example 1 (Identical Centrality Is Not A Necessary Condition). Firstly, we consider an example to show that identical Bonacich centrality for each node is not a necessary condition for a network to be price discrimination free. We consider the example in Figure 1a. In Table 1, we present the adjacency matrix, the centrality vector (based on G and $(G+G^T)$),

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Figure 1 Example networks.

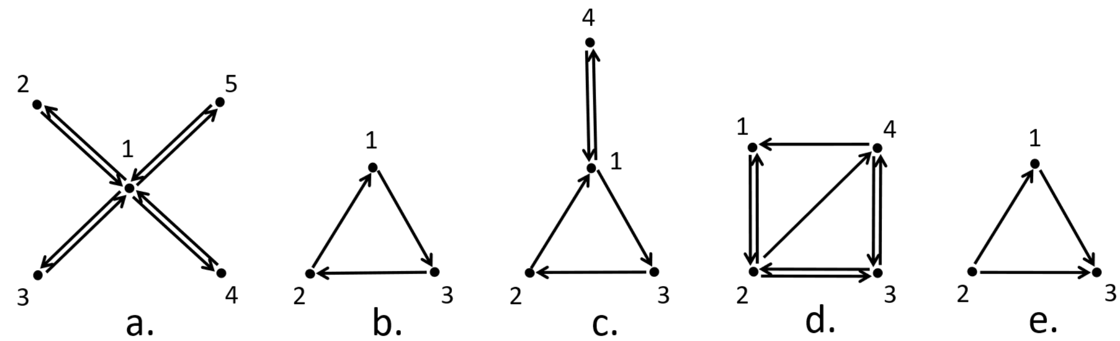

Table 1 Centrality and optimal prices for the network in Figure 1a.

Adjacency Matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|\|G+G^T\|\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|\|G+G^T\|\|}\right)$	Optimal Prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&1&1&1\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}14.7368\\7.6316\\7.6316\\7.6316\\7.6316\end{pmatrix}$	$\begin{pmatrix}14.7368\\7.6316\\7.6316\\7.6316\\7.6316\end{pmatrix}$	$\begin{pmatrix}4.5000\\4.5000\\4.5000\\4.5000\\4.5000\end{pmatrix}$	0

Table 2 Centrality and optimal prices for the network in Figure 1b.

Adjacency Matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|\|G+G^T\|\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|\|G+G^T\|\|}\right)$	Optimal Prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}4.5000\\4.5000\\4.5000\end{pmatrix}$	0

Table 3 Centrality and optimal prices for the network in Figure 1c.

Adjacency Matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|\|G+G^T\|\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|\|G+G^T\|\|}\right)$	Optimal Prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0&1\\0&0&1&0\\1&0&0&0\\1&0&0&0\end{pmatrix}$	$\begin{pmatrix}11.3172\\6.7715\\8.5973\\8.5973\end{pmatrix}$	$\begin{pmatrix}12.7599\\7.9521\\7.9521\\9.5658\end{pmatrix}$	$\begin{pmatrix}4.5000\\3.6931\\5.3069\\4.5000\end{pmatrix}$	0.3683

Table 4 Centrality and optimal prices for the network in Figure 1d.

Adjacency Matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|\|G+G^T\|\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|\|G+G^T\|\|}\right)$	Optimal Prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\0&1&1&0\end{pmatrix}$	$\begin{pmatrix}8.4042\\8.4042\\8.4042\\8.4042\end{pmatrix}$	$\begin{pmatrix}8.1142\\11.2550\\10.2706\\9.7904\end{pmatrix}$	$\begin{pmatrix}5.5782\\3.4218\\4.5000\\4.8459\end{pmatrix}$	0.7267

the optimal prices, and the regret under $\mathbf{p}_0$ for this network. As can be seen, in this network, the centrality of the consumer in the center is different from (larger than) the centrality of other consumers. However, the optimal prices are same for all consumers (this can also be seen by noting

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Table 5 Centrality and optimal prices for the network in Figure 1e.

Adjacency Matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|\|G+G^T\|\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|\|G+G^T\|\|}\right)$	Optimal Prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1.9000\\1.0000\\3.6100\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}4.5000\\0.0000\\9.0000\end{pmatrix}$	2.9363

that the network is symmetric and according to Candogan et al. 2012, symmetric network leads to identical prices). Therefore, identical Bonacich centrality is not a necessary condition for a network to be price discrimination free.

Example 2 (Symmetry Is Not A Necessary Condition). Secondly, we consider an example to show that symmetry is not a necessary condition for a network to be price discrimination free. We consider the example in Figure 1b and we summarize the results for this network in Table 2. It is easily seen that the network is a directed cycle and is not symmetric. However, the optimal prices are the same. Therefore, symmetry is not a necessary condition for a network to be price discrimination free; however, it is a sufficient condition as shown in Candogan et al. (2012).

Example 3 (Same In-Degree and Out-Degree Is Not A Sufficient Condition). Next, we consider an example to show that the in-degree of each node equals its out-degree is not a sufficient condition for a network to be price discrimination free (even for binary networks). We consider the example in Figure 1c. Table 3 presents a summary of results for this network. It is easy to see that for each node in this network, its in-degree and out-degree are the same. However, the optimal prices are different and there is positive value of price discrimination for this network. Therefore, having same in-degree and out-degree for each node in a network is not a sufficient condition for a network to be price discrimination free. (Later we will show that same in-degree and out-degree for each node is a necessary condition.)

Example 4 (Identical G-Based Centrality Is Not A Sufficient Condition). Then, we consider an example to show that all nodes have identical G-based Bonacich centrality (meaning the Bonacich centrality of G) is not a sufficient condition for a network to be price discrimination free. We consider the example in Figure 1d, and we summarize the results for this network in Table 4. As shown in Table 4, each node in this network has the same Bonacich centrality under adjacency matrix G, but the optimal prices are different and there is value of price discrimination for this network. Essentially, having the same G-based Bonacich centrality is equivalent to

$$\mathcal{K}\left(G, \frac{2\rho}{||G+G^T||}\right) = \left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1} = \beta\mathbf{1}$$

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for some constant β. This is further equivalent to

$$\left(\frac{2\rho\beta}{\|G+G^T\|}G\right)\mathbf{1} = (\beta-1)\mathbf{1},$$

which implies that all nodes in the network G have the same in-degree. Therefore, having identical G-based Bonacich centrality or having the same in-degree for each node is not a sufficient condition for a network to be price discrimination free.

Example 5 (Identical $(G+G^T)$-Based Centrality Is Not A Sufficient Condition). Last, we consider an example to show that every node has the same $(G+G^T)$-based Bonacich centrality (meaning the Bonacich centrality of $(G+G^T)$) is not a sufficient condition for a network to be price discrimination free. We consider the example in Figure 1e. Table 5 shows the results for this network. From Table 5, we can see that each node in this network has the same Bonacich centrality based on $(G+G^T)$, but the optimal prices are not the same and there is value of price discrimination for this network. In fact, following similar arguments in Example 4, having the same $(G+G^T)$-based Bonacich centrality is equivalent to having the same degrees for all nodes in the multigraph $(G+G^T)$. Therefore, having identical $(G+G^T)$-based Bonacich centrality, or equivalently, having the same degree for all nodes in the multigraph $(G+G^T)$, is not a sufficient condition for a network to be price discrimination free.

We also point out that, identical centrality based on both G and $(G+G^T)$ is a sufficient condition for a network to be price discrimination free. To see this, we note that having the same G-based Bonacich centrality is equivalent to $G\mathbf{1} = \alpha\mathbf{1}$ for some constant α, and having the same $(G+G^T)$-based Bonacich centrality is equivalent to $(G+G^T)\mathbf{1} = \beta\mathbf{1}$ for some constant β. Combining the two conditions yields $G^T\mathbf{1} = (\beta-\alpha)\mathbf{1}$. Since $\mathbf{1}^TG\mathbf{1} = \mathbf{1}^TG^T\mathbf{1}$, therefore $G\mathbf{1} = G^T\mathbf{1} = \alpha\mathbf{1}$ and $\beta = 2\alpha$. From equation (3.2), having the same $(G+G^T)$-based Bonacich centrality and $G\mathbf{1} = G^T\mathbf{1} = \alpha\mathbf{1}$ together give the same optimal price for all the nodes in the network. Therefore, identical centrality based on both G and $(G+G^T)$ is a sufficient condition for a network to be price discrimination free.

The above examples show that some intuitive condition is not enough to determine whether a network is price discrimination free or not. In the following, we present our main result, which shows that the price discrimination free property is closely related to the balance of walks from and to each node in the network. We have the following theorem.

Theorem 1. *G is a price discrimination free network if and only if $G^k\mathbf{1} = \left(G^T\right)^k\mathbf{1}$ for each positive integer k.*

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The detailed proof of Theorem 1 is provided in Appendix A. Theorem 1 suggests that, G is a price discrimination free network if and only if for any consumer in the network G and any k, the total weight of incoming walks of length k equals the total weight of outgoing walks of the same length. This shows that the price discrimination free property is not a local property (restricted to the characteristics of immediate neighborhood as the symmetry or the same in-degree and out-degree properties are), but a global property. It also shows that same in-degree and out-degree for each node is a necessary but not a sufficient condition for a network to be a price discrimination free network. However, when the network $(G+G^T)$ is regular, this condition is also sufficient as shown in the following corollary.

COROLLARY 1. *If $(G+G^T)$ is a regular network, then G is a price discrimination free network if and only if $G\mathbf{1}=G^T\mathbf{1}$, or the in-degree equals the out-degree for each node.*

The corollary immediately follows from Theorem 1. When $(G+G^T)$ is regular, for each node in G, if the in-degree and out-degree are equal, then the total weight of incoming walks of any arbitrary length k equals the total weight of outgoing walks of the same length (thus satisfying the condition for Theorem 1). Corollary 1 provides a simple test for regular networks. However, it may be difficult to obtain such simple tests for general networks. Using a test based upon evaluating the weights of smaller walks, a bound on the fractional regret may be obtained as a function of the positive network externality coefficient ρ. For example, if for each node and each $k \leq K$, the total weight of incoming walks of length k is equal to the total weight of outgoing walks of the same length for the node, i.e., $G^k\mathbf{1}=(G^T)^k\mathbf{1}$ for each $k \leq K$, then the fractional regret $R_F(\mathbf{p}_0)$ is bounded by ρ^{2K+1}. The statement follows from the proof of Theorem 1 in Appendix A. A family of tests could help characterize networks with "small" value of price discrimination. Such a family of tests is beyond the scope of this paper and is an interesting future direction of research.

5. Value of Price Discrimination for Erdös-Renyi Random Networks

In Section 4, we study conditions under which there exists value of price discrimination for a deterministic network. In this section, we study the value of price discrimination in random networks. We conduct an asymptotic analysis. That is, we consider a sequence of networks indexed by n. In the nth network, there are n consumers embedded in a random network $G(n)$ and for each network $G(n)$, we consider the same pricing problem as described in Section 3. We assume a, c and ρ are the same for each pricing problem in the sequence. Specifically, the utility for consumer i in the nth network is

$$u_i(x_i(n),\mathbf{x}_{-i}(n),p_i(n)) = ax_i(n) - x_i(n)^2 + 4\rho x_i(n)\sum_{j\neq i}\frac{G_{ij}(n)}{\|G(n)+G(n)^T\|}x_j(n) - p_i(n)x_i(n), \quad (5.1)$$

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where $\mathbf{x}(n)$ is the vector representing the consumption profile and $\mathbf{p}(n)$ is the price vector in the nth pricing problem. We denote $\mathbf{x}^*(n, \mathbf{p}(n))$ as the consumption equilibrium, $\mathbf{p}^*(n)$ as the optimal price vector, $\pi^*(n)$ as the optimal profit, $\mathbf{p}_0(n)$ as the optimal uniform price vector,[5] $\pi_0(n)$ as the profit under optimal uniform pricing, $R(\mathbf{p}_0(n))$ as the monopolist's regret under optimal uniform pricing and $R_F(\mathbf{p}_0(n))$ as the monopolist's fractional regret under optimal uniform pricing in the nth pricing problem.

We focus on a special class of random networks — the directed Erdös-Renyi network. A directed Erdös-Renyi network $G(n, p(n))$ is a directed, binary random network with n nodes, where links between ordered pairs of nodes exist independently with a probability $p(n)$ (we allow $p(n)$ to be a function of n). More precisely, the adjacency matrix of a directed Erdös-Renyi network $G(n, p(n))$ is a random binary matrix satisfying

$$G_{ij}(n, p(n)) = \begin{cases} 1, & \text{with probability } p(n) \\ 0, & \text{with probability } 1 - p(n) \end{cases}$$

for $i \neq j \in \{1, 2, ..., n\}$, and $G_{ii}(n, p(n)) = 0$ for $i \in \{1, 2, ..., n\}$. We note that the sample networks $G(n, p(n))$ are usually not symmetric. We call $p(n)$ the influence probability among consumers, it also represents the expected density of the random network.

Given that the networks are random, the performance metrics, including the monopolist's regret and fractional regret under optimal uniform pricing, are random. We are interested in the asymptotic properties of the performance metrics. In particular, we are interested in the expected value of these random performance metrics as n grows large.

5.1. Main Results

We now introduce our main results. A sketch of the proofs is presented in Section 5.2 with the complete proofs provided in the appendix.

Our first result shows that for any network density sequence $(p(n))_{n\in\mathbb{N}}$, the monopolist's expected fractional regret vanishes asymptotically. For simplicity of representation, we will represent the sequence $(p(n))_{n\in\mathbb{N}}$ as $p(n)$, as needed. We have the following theorem.

THEOREM 2. *Given any sequence of network densities $p(n)$, for the sequence of Erdös-Renyi random networks, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = o(1)$.*

[5] The optimal uniform price for each consumer is $\frac{a+c}{2}$. The optimal uniform price vectors for different n differ only in the dimension of the vectors.

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Theorem 2 shows that for Erdös-Renyi networks, the expected regret grows sub-linearly in the size of the network, and the expected fractional regret vanishes asymptotically. This implies that the value of price discrimination is negligible asymptotically, and the optimal uniform pricing is good enough to guarantee almost all of the monopolist's profit when the size of the network grows large enough, irrespective of the network density. We note that the optimal uniform price $\mathbf{p}_0(n) = \frac{a+c}{2}\mathbf{1}$ is independent of the network structure. This implies that the monopolist does not even need to invest effort to learn the underlying social networks in the asymptotic regime. We provide the detailed proof of Theorem 2 in Appendix D.

Theorem 2 is powerful because it does not put any constraints on the sequence $p(n)$, but as a result the uniform upper bound on the expected regret and expected fractional regret is loose. In the following, we present tighter asymptotic bounds of expected regret and expected fractional regret for specific ranges of $p(n)$. Note that asymptotically, Erdös-Renyi networks are empty almost surely if $p(n) = O(n^{-2})$ (see Erdos and Rényi 1960). In this range of network densities, there is no network effect and thus no value of price discrimination. Therefore, in the following discussions, we are only interested in the range where $p(n) = \omega(n^{-2})$. Specifically, we consider three different cases: (i) expected in/out degree vanishes asymptotically, i.e., $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ (in the random graph literature, such networks are often called *very sparse* networks); (ii) expected in/out degree stays asymptotically bounded and positive, i.e., $p(n) = \Theta(n^{-1})$ (in the random graph literature, such networks are often called *critically sparse* networks); and (iii) expected in/out degree asymptotically grows faster than $\log n$, i.e., $p(n) = \omega\left(\frac{\log n}{n}\right)$ (in the random graph literature, such networks are often called *dense* networks).

We first consider the case of very sparse networks where $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$. We have the following results.

THEOREM 3. *When $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(n^2 p(n)\right)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta\left(np(n)\right)$.*

Now we provide some explanations for the results in Theorem 3. Probabilistically, the networks in this case are acyclic, extremely sparse and fragmented. In such cases, the effect of any consumer's purchase is restricted to the consumers in the same component in the network. With increasing network density, the size of the components grows and a central consumer is able to influence a larger set of consumers without being influenced by them. Due to this growing imbalance, with increasing network density in this range, the expected regret and expected fractional regret increase. The detailed proof of this case is given in Appendix D.

Next we consider the case of critically sparse networks, where $p(n) = \Theta(n^{-1})$.

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THEOREM 4. *When* $p(n) = \Theta(n^{-1})$, *the expected regret* $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(\frac{\log\log n}{\log n} n\right)$, *and the expected fractional regret* $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta\left(\frac{\log\log n}{\log n}\right)$.

In this range of $p(n)$, a sharp phase transition happens. According to Janson et al. (1993), as $p(n)$ increases from $\frac{1}{n} - O\left(n^{-\frac{4}{3}}\right)$ to $\frac{1}{n} + O\left(n^{-\frac{4}{3}}\right)$, small components merge to form a giant component containing a positive fraction of consumers in the network and cycles emerge. While the emergence of giant component increases the value of price discrimination, the emergence of cycles reduces this value. However, there are not enough cycles in the network to balance the influence and the effects of giant components and longer paths dominate. Therefore in this range of $p(n)$, the value of price discrimination is higher than the case when $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$. Again, the detailed proof is given in Appendix D.

Finally, we consider the case of denser networks, where $p(n) = \omega\left(\frac{\log n}{n}\right)$. We have the following results.

THEOREM 5. *When* $p(n) = \omega\left(\frac{\log n}{n}\right)$, *the expected regret* $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(p(n)^{-1}\right)$, *and the expected fractional regret* $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta\left(n^{-1}p(n)^{-1}\right)$.

In this range of $p(n)$, the network is connected with high probability, contains many cycles and is asymptotically regular and balanced. The in/out degree distribution is tightly concentrated around the average degree and converges asymptotically to a normal distribution. Therefore, networks in this range of densities have very small value of price discrimination. Furthermore, as the network gets denser, the value of price discrimination decreases. This is because the coefficient of variation of the in/out degree distribution becomes smaller, leading to a more balanced network. The detailed proof is given in Appendix D.

In summary, we found that under different ranges of $p(n)$, the expected regret and expected fractional regret under the optimal uniform pricing strategy follow different rates. When $p(n)$ is relatively small or relatively large, there is not much value of price discrimination. For sparser networks, the value of price discrimination increases in density and for denser networks, the value of price discrimination decreases in density. The value of price discrimination reaches its maximum in the range where the average degree $np(n)$ is asymptotically bounded away from zero and is growing slower than the logarithm of n. We summarize our main results in Table 6.

5.2. Proof Concepts

While the proofs of the main results are quite complicated and are presented in the appendix, we provide a sketch of those proofs and intermediate results in this section. Random networks demonstrate very different properties for different densities and therefore the techniques to quantify

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Table 6 The value of price discrimination for different ranges of network densities.

Network Density $p(n)$	Expected Regret $\mathbf{E}_G[R(\mathbf{p}_0)]$	Expected Fractional Regret $\mathbf{E}_G[R_F(\mathbf{p}_0)]$
$O\left(n^{-(1+\epsilon)}\right)$	$\Theta\left(n^2 p(n)\right)$	$\Theta\left(np(n)\right)$
$\Theta\left(n^{-1}\right)$	$\Theta\left(\frac{\log\log n}{\log n} n\right)$	$\Theta\left(\frac{\log\log n}{\log n}\right)$
$\omega\left(\frac{\log n}{n}\right)$	$\Theta\left(p(n)^{-1}\right)$	$\Theta\left(n^{-1}p(n)^{-1}\right)$

the rates of regret/fractional regret are different for different ranges of network densities, even though an overarching theme emerges in the proofs. The proof of Theorem 2 uses a combination of techniques used to prove Theorems 3, 4, and 5 with some additional complexity, so we will focus on the sketch of the proofs of Theorems 3, 4, and 5 in this section.

Overall, our proof technique relies on decomposing the profit and regret into components corresponding to walks of different lengths and then estimating each component. The following is the outline for the rest of this section.

1. In Section 5.2.1, we introduce two important quantities used in the proofs, namely the profit contribution from walks of different lengths and the regret contribution from walks of different lengths. We also demonstrate how these quantities behave for different network densities. Then based on these concepts, we give a high level summary of the proof ideas.

2. In Section 5.2.2, we introduce the techniques for obtaining the upper bounds on the expected regrets. To quantify the profit contributions and regret contributions from walks of different lengths, we need to characterize the asymptotic behavior of the largest eigenvalue of the multigraph $(G(n)+G(n)^T)$, and the number of walks of different lengths in the multigraph $(G(n)+G(n)^T)$ and network $G(n)$. We build upon literature on the spectra of Erdös-Renyi random networks to characterize the asymptotic behavior of the largest eigenvalue of $(G(n)+G(n)^T)$. We then introduce novel techniques to quantify the number of walks of different lengths in random networks.

3. In Section 5.2.3, we introduce our approach for obtaining matching lower bounds. Our approach relies on counting short walks and small components in random networks.

5.2.1. Profit Contribution and Regret Contribution

Before introducing the definition of profit contribution and regret contribution, we first define the concept of value of network effect.

Definition 2 (Value of Network Effect). The value of network effect for the monopolist is the additional expected profit the monopolist can generate under optimal pricing due to the

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presence of network externalities. More specifically,

$$\mathrm{VoN}(n) = \mathbf{E}_G[\pi^*(n)] - \frac{n}{2}\left(\frac{a-c}{2}\right)^2 = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=1}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1}^T \left(G(n)+G(n)^T\right)^k \mathbf{1}\right]. \tag{5.2}$$

In the definition, $\frac{n}{2}\left(\frac{a-c}{2}\right)^2$ is the optimal profit when there is no network effect, which is obtained by substituting $G(n)$ with an all-zero matrix in the optimal profit equation (3.3). The value of network effect is a natural upper bound on the expected regret. It turns out, as we show in the proof of Theorem 3, that this upper bound is tight for sparse networks as it matches the lower bound. According to equation (5.2), the value of network effect can be decomposed into the profit contributions from walks of different lengths. The *profit contribution from walks of length* k is the kth term in the series represented in equation (5.2), which is proportional to the number of walks of length k when controlling for the spectral norm of the multigraph $(G(n)+G(n)^T)$. We note that by the definition of spectral norm, for any non-empty network $G(n)$ and any $k>0$,

$$0 < \mathbf{1}^T \left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1} \leq n, \tag{5.3}$$

and the highest value of $\mathbf{1}^T \left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1}$, for any given k, is attained when $(G(n)+G(n)^T)$ is regular (all row sums are equal).[6] Therefore, the value of network effect defined in equation (5.2) satisfies the following inequality:

$$0 \leq \mathrm{VoN}(n) \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=1}^{\infty} \rho^k n = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{n\rho}{1-\rho}. \tag{5.4}$$

Correspondingly, *the regret contribution from walks of length* k is the difference between the profit contribution from walks of length k and the expected value of the kth term in the series represented in equation (3.4), or

$$\frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k \left(\mathbf{1}^T \left(G(n)+G(n)^T\right)^k \mathbf{1} - 2^k \mathbf{1}^T G(n)^k \mathbf{1}\right)\right]. \tag{5.5}$$

The profit contribution and the regret contribution from longer walks asymptotically decay exponentially because $\rho<1$. Most of the value of network effect and the regret come from contributions from shorter walks. For better illustration of the profit contribution and regret contribution from walks of different lengths, we provide some numerical results in Figure 2. These results are obtained

[6] To see this result, we note that $\|G(n)+G(n)^T\|^k = \|(G(n)+G(n)^T)^k\| = \max_\xi \frac{\xi^T (G(n)+G(n)^T)^k \xi}{\xi^T \xi} \geq \frac{\mathbf{1}^T (G(n)+G(n)^T)^k \mathbf{1}}{\mathbf{1}^T \mathbf{1}}$. Therefore $\mathbf{1}^T \left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1} \leq \mathbf{1}^T \mathbf{1} = n$, and the equality holds only when $\mathbf{1}$ is the largest eigenvector of $(G(n)+G(n)^T)^k$. This is equivalent to that $(G(n)+G(n)^T)$ is a regular graph.

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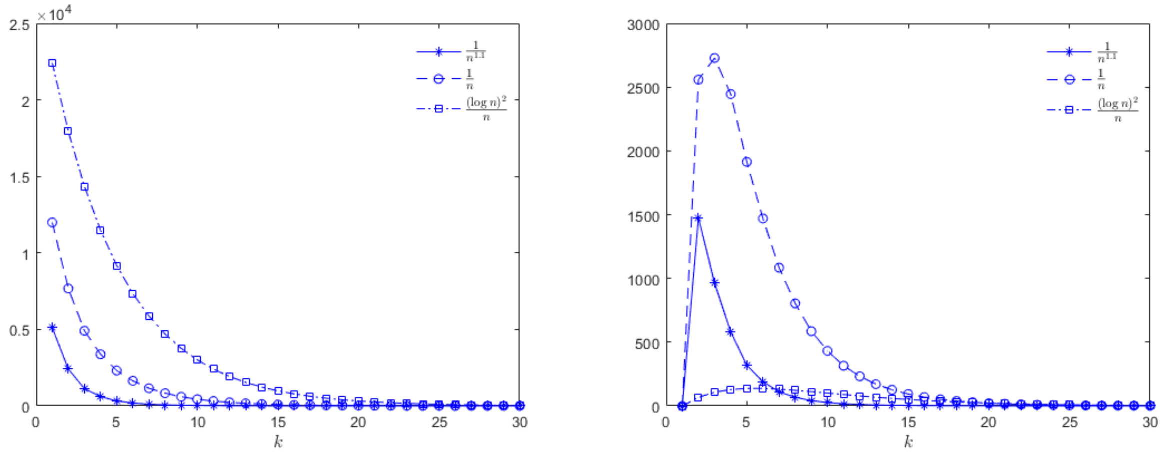
Figure 2 Average profit contribution and regret contribution from walks of different lengths in Erdös-Renyi networks ($n = 100000$, $\rho = 0.8$)



(a) Average profit contribution

(b) Average regret contribution

by generating $N = 100$ independent directed Erdös-Renyi networks with $n = 100,000$ nodes under each $p(n) \in \{\frac{1}{n^{1.1}}, \frac{1}{n}, \frac{(\log n)^2}{n}\}$. We choose $a = 2$, $c = 0.5$, and $\rho = 0.8$ in the numerical experiments. For each realization of $G(n, p(n))$ and each k, we compute the corresponding profit contribution and regret contribution from walks of length k, and plot the average values in Figure 2. From Figure 2, the profit contribution from walks of length k decays as k increases for all network densities; and for any given k, the profit contribution from walks of length k increases with network density. The regret contribution from walks of length k is unimodular. For $k = 0$ and 1, the regret contribution is 0. As k increases, the regret contribution from walks of length k initially increases. This is because the expected number of walks of length k in the multigraph $(G(n) + G(n)^T)$ increases faster than the 2^k times the expected number of walks in the network $G(n)$ due to imbalance in the network. However, as k passes a certain threshold, the regret contribution from walks of length k starts decreasing in k. Furthermore, the location of the peak increases with the density of the network, suggesting that when quantifying (approximating) the expected regret of denser networks, regret contributions from longer walks should be considered. We also observe that when we move from very sparse networks to critically sparse networks, the peak value increases (because there are increasing number of longer walks while there are not many cycles); but when we move further to denser networks, the peak value decreases sharply (because there are increasing number of cycles). The observations are consistent with the results in Theorems 3, 4, and 5 and will be formalized in the proofs.

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With the above definitions, we provide some high-level proof ideas. First, we note that the profit contribution from walks of any length k dominates the regret contribution from walks of length k. Therefore, the value of network effect is a natural upper bound on the expected regret. For very sparse networks discussed in Theorem 3, the value of network effect turns out to be of the same order as the expected regret and therefore the bound is tight. For critically sparse networks discussed in Theorem 4, the regret contribution from walks of length 1 is zero. Therefore, we use the total profit contribution from walks of length greater than 1 as the upper bound on the expected regret. It turns out that the bound is also tight in this range because we are able to find a matching lower bound.

For denser networks in Theorem 5, the profit contributions from longer walks could be much larger than the regret contributions. We therefore quantify the expected regret directly. In particular, we decompose the expected regret into a finite sum (of regret contributions from walks of length up to $\sqrt{\log(np(n))}$ or the square root of the logarithm of the average degree) and a tail term. We obtain upper bounds on both the finite sum and the tail term. Then we provide a matching lower bound on the expected regret and prove the bound is tight.

5.2.2. Deriving Upper Bounds on Expected Regrets

Despite the above observations, the value of network effect and the expected regret expressions are still very complicated. In particular, the kth term in equation (5.2) and equation (5.5) are expectations of the ratio of two random variables (number of walks of length k and the spectral norm of the multigraph $(G(n)+G(n)^T)$ raised to the power k). To overcome this difficulty, we first obtain asymptotic properties for the spectral norm.

Then we count the expected number of walks of any length k in multigraph $(G(n)+G(n)^T)$ to quantify the profit contribution from walks of length k. To prove Theorem 5, we also need to obtain the expected difference in the number of walks of length k in multigraph $(G(n)+G(n)^T)$ and 2^k times the number of walks of length k in network $G(n)$ to quantify the regret contribution from walks of length k. We present the results of these two parts in the following.

1) Asymptotic Spectral Norm of the Multigraph $(G(n)+G(n)^T)$

The asymptotic spectral norm of $(G(n)+G(n)^T)$ can be derived from the spectral property of undirected Erdös-Renyi networks. For undirected Erdös-Renyi networks, Krivelevich and Sudakov (2003) show that the largest eigenvalues are highly concentrated. Moreover, the largest eigenvalue demonstrates phase transitions for different ranges of $p(n)$. Specifically, Krivelevich and Sudakov (2003) show the following theorem and two following corollaries. In the following, we say a sequence of events E_n hold *asymptotically almost surely (a.a.s.)* if the probabilities of E_n converge to 1.

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Theorem 6 **(Krivelevich and Sudakov 2003)**. *Let $G(n)$ be an undirected Erdös-Renyi network. If Δ is the maximum degree of $G(n)$, then almost surely the largest eigenvalue of the adjacency matrix of $G(n)$ satisfies $\|G(n)\| = (1+o(1))\max\{\sqrt{\Delta}, np(n)\}$, where the $o(1)$ term tends to zero as $\max\{\sqrt{\Delta}, np(n)\}$ tends to infinity.*

Therefore, for large undirected Erdös-Renyi networks, the spectral norm is almost surely of the same order as the maximum between the square root of the largest degree and the average degree. While Δ is a random variable given that $G(n)$ is a random network, one can use convergence results on Δ to obtain the asymptotic property for $\|G(n)\|$. The following corollary provided in Krivelevich and Sudakov (2003) is a result about the convergence of the largest degree Δ when $p = \Theta(n^{-1})$.

Corollary 2 **(Krivelevich and Sudakov 2003)**. *Let $G(n)$ be an undirected Erdös-Renyi network. When $p(n) = \Theta(n^{-1})$, almost surely $\|G(n)\| = (1+o(1))\sqrt{\frac{\log n}{\log\log n}}$.*

Krivelevich and Sudakov (2003) omit the technical details of the proof of the Corollary 2. The proof uses the fact that when $p(n) = \Theta(n^{-1})$, the binomial degree distribution of a node can be approximated by a Poisson degree distribution. Then the largest degree Δ is determined by the maximum of a set of i.i.d. Poisson random variables, which converges to $\frac{\log n}{\log\log n}$ as shown in Kimber (1983).

The following corollary shows that when $p(n)$ grows larger, the average degree dominates the square root of the largest degree, and the spectral norm is of the same order as the average degree.

Corollary 3 **(Krivelevich and Sudakov 2003)**. *Let $G(n)$ be an undirected Erdös-Renyi network. When $p(n) \geq \frac{\sqrt{\log n}}{n}$, almost surely $\|G(n)\| = (1+o(1))np(n)$.*

Krivelevich and Sudakov (2003) did not provide the spectral norm of $G(n)$ for very sparse networks. In the following, we derive a bound on the spectral norm for this setting. In particular, we consider $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ The following lemma shows that in this case, the spectral norm is asymptotically almost surely bounded by some constant.

Lemma 1. *Let $G(n)$ be an undirected Erdös-Renyi network. When $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, asymptotically almost surely $1 \leq \|G(n)\| \leq m(\epsilon)$, where $m(\epsilon)$ is a constant that only depends on ϵ.*

Lemma 1 shows that, when the network is very sparse, i.e., $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, the spectral norm does not grow in the size of the networks. This is different

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from the spectral property of denser networks (for example, the networks specified in Corollaries 2 and 3), where the spectral norms grow in the size of the networks.

Now we have the rate at which the spectral norm of the undirected Erdös-Renyi networks grows for different network densities. Remember that $(G(n)+G(n)^T)$ represents an undirected network. However, the elements in the matrix $(G(n)+G(n)^T)$ can take values from $\{0,1,2\}$ and it is not a binary network. Therefore we still need some extra efforts to obtain the bound we need.

To establish the relation between the spectral norm of $G(n)$ and $(G(n)+G(n)^T)$, we decompose the directed Erdös-Renyi network adjacency matrix $G(n)$ into the sum of an upper triangle matrix $G_1(n)$ and a lower triangle matrix $G_2(n)$, i.e. $G(n)=G_1(n)+G_2(n)$. Then $(G_1(n)+G_1(n)^T)$ and $(G_2(n)+G_2(n)^T)$ are two independent undirected Erdös-Renyi networks with probability $p(n)$, and $(G(n)+G(n)^T)$ can be viewed as the sum of two independent undirected Erdös-Renyi networks with probability $p(n)$. By the property of spectral norm, we have

$$\|G(n)+G(n)^T\| \leq \ \|G_1(n)+G_1(n)^T\|+\|G_2(n)+G_2(n)^T\|.$$

Since $(G_1(n)+G_1(n)^T)$ is a non-negative matrix, by Perron-Frobenius Theorem, there is a non-negative eigenvector associated with its largest eigenvalue. Using the property of the largest eigenvalue, we can show that

$$\|G_1(n)+G_1(n)^T\| \leq \|G(n)+G(n)^T\|.$$

By Theorem 6 and the above analysis, $\|G(n)+G(n)^T\| \in [(1+o(1))\max\{\sqrt{\Delta}, np(n)\}, (2+o(1))\max\{\sqrt{\Delta}, np(n)\}]$ almost surely, where Δ is the maximum degree of an undirected Erdös-Renyi network with influence probability $p(n)$. Thus, asymptotically $\|G(n)+G(n)^T\|$ is of the same order of the largest eigenvalue of the undirected Erdös-Renyi networks with the same probability $p(n)$.

Now we derive an alternative lower bound of $\|G(n)+G(n)^T\|$ to obtain a sharper characterization of $\|G(n)+G(n)^T\|$ when $p(n)=\omega\left(\frac{\log n}{n}\right)$. We use $|E(G(n)+G(n)^T)|$ to denote the number of edges in multigraph $(G(n)+G(n)^T)$. When $p(n)=\omega(n^{-2})$, the number of edges in $(G_1(n)+G_1(n)^T)$ (or $(G_2(n)+G_2(n)^T)$) is $(1+o(1))\frac{n^2p(n)}{2}$ almost surely (Krivelevich and Sudakov 2003), therefore $|E(G(n)+G(n)^T)|=(2+o(1))\frac{n^2p(n)}{2}$ almost surely. Also since

$$\|G(n)+G(n)^T\| = \max_{\xi} \frac{\xi^T(G(n)+G(n)^T)\xi}{\xi^T\xi} \geq \frac{\mathbf{1}^T(G(n)+G(n)^T)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2|E(G(n)+G(n)^T)|}{n},$$

therefore almost surely,

$$\|G(n)+G(n)^T\| \geq (2+o(1))np(n).$$

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To summarize, although $\|G(n)+G(n)^T\|$ is a random variable, we have the following asymptotic characterization of $\|G(n)+G(n)^T\|$ for different ranges of values of $p(n)$:

$$\|G(n)+G(n)^T\| = \begin{cases} \Theta(1) & \text{, if } p(n) = O\left(n^{-(1+\epsilon)}\right) \text{ for } \epsilon > 0 \\ \Theta\left(\sqrt{\dfrac{\log n}{\log\log n}}\right) & \text{, if } p(n) = \Theta(n^{-1}) \\ (2+o(1))np(n) & \text{, if } p(n) = \omega\left(\frac{\log n}{n}\right) \end{cases} \quad \text{almost surely.} \tag{5.6}$$

2) Expected Number of Walks of Different Lengths

We now quantify the expected number of walks of length k in $(G(n)+G(n)^T)$. We start with the case $k=2$. We have the following lemma.

LEMMA 2. *For directed Erdös-Renyi network $G(n)$,*

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^2\mathbf{1}\right] = 2n(n-1)p(n)(1-3p(n)+2np(n)).$$

Note that Lemma 2 provides an exact calculation of the expected number of walks of length 2 in the multigraph $(G(n)+G(n)^T)$. The proof of Lemma 2 is based on considering the degree distribution of a neighboring node and is given in Appendix C. For walks with longer lengths, i.e., $k \geq 3$, it is difficult to consider all possible repeated links and thus an exact calculation of walks is difficult. However, we observe that the number of walks of length k in the multigraph $(G(n)+G(n)^T)$ can be upper bounded by the product of the number of walks of length t for any $0 \leq t \leq k$ and the spectral norm of $(G(n)+G(n)^T)$ raised to the power $k-t$. The result is given in the following lemma.

LEMMA 3. *For any adjacency matrix $G(n)$, given a positive integer t, for any integer $k \geq t$,*

$$\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1} \leq \|G(n)+G(n)^T\|^{k-t}\mathbf{1}^T\left(G(n)+G(n)^T\right)^t\mathbf{1}.$$

Lemmas 2 and 3 help obtain an upper bound on the expected number of walks of any length k asymptotically in the multigraph $(G(n)+G(n)^T)$, because the spectral norm converges asymptotically almost surely according to equation (5.6). This bound helps us obtain a bound on the profit contribution from walks of different lengths and in turn the value of network effect in very sparse networks (in Theorem 3) and the total profit contribution from all walks of length greater than 1 in critically sparse networks (in Theorem 4).

For denser networks, the profit contribution from walks of length greater than 1 is significant and the value of network effect is large. Therefore, we need to quantify the regret contribution from walks of different lengths. To quantify the regret contribution from walks of different lengths, we need the expected difference in the number of walks of length k in the multigraph $(G(n)+G(n)^T)$

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and 2^k times the number of walks of length k in network $G(n)$. Unfortunately, an exact computation of the number of walks of any given length is difficult. However, we can provide a lower bound for the expected number of walks of different lengths in a directed Erdös-Renyi network $G(n)$ by ignoring the possibly repeated links. We have the following lemma.

LEMMA 4. *For directed Erdös-Renyi network $G(n)$, for any integer $k \geq 2$,*

$$\mathbf{E}_G(\mathbf{1}^T G(n)^k \mathbf{1}) \geq n^2(n-1)^{k-2}(n-2)p(n)^k.$$

The following proposition provides bounds on the differences between $\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right]$ and $2^k\mathbf{E}_G\left[\mathbf{1}^T G(n)^k \mathbf{1}\right]$ for different k.

PROPOSITION 1. *For any directed Erdös-Renyi network $G(n)$, the following statements hold.*
(i) For any integer $k \geq 0$,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right] \geq 2^k\mathbf{E}_G\left[\mathbf{1}^T G(n)^k \mathbf{1}\right].$$

(ii) For $0 \leq k < n$ and $\frac{1}{n-k+1} \leq p(n) < 1$,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right] - 2^k\mathbf{E}_G\left[\mathbf{1}^T G(n)^k \mathbf{1}\right] \leq (k-1)2^{k-1}P(n,k)p(n)^{k-1} + (k-1)^k 2^k P(n,k-1)p(n)^{k-2},$$

where $P(n,k) = \frac{n!}{(n-k)!}$ represents the number of ways of permuting k out of n objects.

The proof of Proposition 1 is quite involved. In the proof, we develop novel counting techniques using the concepts of *graph motifs*. We refer the readers to Appendix C for the detailed proof. We note that Proposition 1 (i) only holds in expectation. In fact, there exists G for which $\mathbf{1}^T\left(G+G^T\right)^k\mathbf{1} < 2^k\mathbf{1}^T G^k \mathbf{1}$. An example of such G is

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In this case, $\mathbf{1}^T(G+G^T)^3\mathbf{1} = 134$ and $2^3\mathbf{1}^T G^3 \mathbf{1} = 136$, and therefore $\mathbf{1}^T(G+G^T)^k\mathbf{1} < 2^k\mathbf{1}^T G^k \mathbf{1}$. Also, in Proposition 1 (ii), we only bound the expected difference in the number of walks of length k in multigraph $(G(n)+G(n)^T)$ and 2^k times the number of walks of length k in network $G(n)$ for $k \leq n+1-\frac{1}{p(n)}$. For dense networks and larger k, we use a concentration inequality on the in/out degree of the consumers in the network to obtain a bound on the tail profit contribution.

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In particular, we show that when the network density is large enough, i.e., when $p(n) = \omega\left(\frac{\log n}{n}\right)$, the in/out degree of every node is highly concentrated around the average degree. We have the following proposition.

PROPOSITION 2. *Let $c(n)$ be a function of n such that $\lim_{n\to\infty} c(n) = +\infty$ and $c(n)\log n < n$ for all n. Let $\delta(n)$ be another function of n such that $\delta(n) = \Theta(\frac{1}{\sqrt{c(n)}})$ and $\sqrt{\frac{12}{c(n)}} < \delta(n) < 1$. If $p(n) = \frac{c(n)\log n}{n}$, then almost surely every node of the directed Erdös-Renyi network $G(n)$ has in/out degree in the range of $[(1-\delta(n))c(n)\log n, (1+\delta(n))c(n)\log n]$.*

5.2.3. Deriving Lower Bounds on Expected Regrets and Expected Fractional Regrets

With the above results, we are able to obtain upper bounds on the regrets needed for Theorems 3, 4 and 5. To obtain matching lower bounds, we use different methods for sparse and dense networks. For very sparse and critically sparse networks in the range in Theorems 3 and 4, matching lower bounds of the regret can be obtained by counting the expected number of components in the network that consist of two nodes with exactly one directed link (see Appendix B for details). For denser networks in the range in Theorem 5, a lower bound on the expected regret can be obtained by the regret contribution from walks of length 2. We refer the detailed analysis of this part to Appendix D.

Finally, in order to obtain bounds on the expected fractional regret, we also need to show that the optimal profit is of order $\Theta(n)$. By equation (3.3),

$$\pi^*(n) = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=0}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1}^T\left(G(n)+G(n)^T\right)^k \mathbf{1}\right].$$

Considering only $k=0$ in the above summation, we get the inequality $\pi^*(n) \geq \frac{n}{2}\left(\frac{a-c}{2}\right)^2$. Using equation (5.3), we get that $\pi^*(n) \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=0}^{\infty} \rho^k n = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{n}{1-\rho}$. Combining the lower and upper bounds of the optimal profit, we have

$$\pi^*(n) = \Theta(n). \tag{5.7}$$

The results on the fractional regret are thus established.

6. Value of Price Discrimination for Random Networks with General Degree Distributions

In this section, we extend our analysis to more general random networks. We first establish a result to obtain upper bounds on the value of price discrimination in general networks. Then we apply the result to obtain upper bounds on the value of price discrimination for power-law networks.

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The following theorem provides a general framework to obtain an upper bound on the expected regret and expected fractional regret in random networks with general degree distributions. The proof is provided in Appendix E.

THEOREM 7. *For any sequence of directed, integer-valued random networks $G(n)$, if $(G(n) + G(n)^T)$ has degree distribution $d(n) \sim F(n)$ and the network satisfies:*

1. $\max_{1 \le i \le n} \sum_{j=1}^{n} (G(n)_{ij} + G(n)_{ji}) \ge \xi(n)$ *asymptotically almost surely for some sequence* $\xi(n)$*;*
2. $\mathbf{E}[d(n)^2] \le \gamma(n)$ *for some sequence* $\gamma(n)$*;*

then the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = O\left(\frac{n\gamma(n)}{\xi(n)}\right)$*, and the expected fractional regret* $\mathbf{E}_G[R_F(\mathbf{p}_0)] = O\left(\frac{\gamma(n)}{\xi(n)}\right)$*.*

Theorem 7 provides a general framework to evaluate the value of price discrimination in random networks by using the asymptotic behavior of the ratio of the second moment of the degree distribution and the maximum degree. Thus, if we can obtain a lower bound on the maximum degree of $(G(n) + G(n)^T)$ and an upper bound on the second moment of the degree distribution of $(G(n) + G(n)^T)$, then we can obtain an upper bound on the expected regret and expected fractional regret.

In the following, we apply Theorem 7 to obtain bounds on the regret and fractional regret for an important class of random networks — the power-law networks. The power-law networks, also called the scale-free networks, is a class of networks whose degree distribution follows a power-law. For comprehensive discussions of the power-law networks, we refer the readers to Aiello et al. (2001).

We consider the continuous version of the power-law distributions with p.d.f. $f(x) \propto x^{-\alpha}$ and c.d.f. $F(x)$ such that $1 - F(x) \propto x^{1-\alpha}$, where α is called the exponent of the power-law distribution. The exponent α of the power-law distribution has a significant impact on the properties of the distribution. We are interested in the range $\alpha > 2$ because the expected degree diverges for $\alpha \le 2$. The range of α of interest also covers the $\alpha = 3$ case obtained from preferential attachment process originally shown in Barabási and Albert (1999). Power-law networks also demonstrate a structural cutoff because the maximum degree in a finite network is not unbounded. In Newman (2003), the authors show that for networks of size n with power-law distributions following $p_d \sim d^{-\alpha}$, the expected maximum degree (or the structural cutoff) follows $d_{max} \sim n^{\frac{1}{\alpha-1}}$, which is of order $o(n)$ given that $\alpha > 2$. We use a more conservative cutoff and impose an upper bound n for the support of the degree distribution. This upper bound on the degree is always satisfied for binary networks because the in/out degree of any node in $G(n)$ is at most n. To obtain a valid probability

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distribution, we further impose a lower bound $x_{min} \geq 1$ on the support of the distribution. A lower bound on the support is also commonly used in power-law degree distributions to obtain valid distributions. In the preferential attachment process, the number of edges for every newborn node provides the lower bound on the degree (Barabási and Albert 1999). With the above considerations, the p.d.f. of the power-law distribution we consider follows

$$f(x) = \frac{\alpha - 1}{x_{min}} \left(\frac{x}{x_{min}}\right)^{-\alpha} \Bigg/ \left(1 - \left(\frac{n}{x_{min}}\right)^{1-\alpha}\right) \quad \text{for } x_{min} \leq x \leq n, \tag{6.1}$$

and the c.d.f. of the distribution follows

$$F(x) = P(X \leq x) = \left(1 - \left(\frac{x}{x_{min}}\right)^{1-\alpha}\right) \Bigg/ \left(1 - \left(\frac{n}{x_{min}}\right)^{1-\alpha}\right) \quad \text{for } x_{min} \leq x \leq n. \tag{6.2}$$

When we consider a sequence of power-law distributions in terms of n, we use $F(n)$ to denote the sequence of c.d.f. of the distributions. In the following, we consider random networks $G(n)$ whose in-degrees $a_1, ..., a_n$ are i.i.d. with distribution $F(n)$, and out-degrees $b_1, ..., b_n$ are also i.i.d. with distribution $F(n)$. We allow pairwise correlation between the in-degree and out-degree of a node, i.e. $\mathbf{corr}(a_i, b_i) = \rho_{a,b}$ and $\rho_{a,b} \in [-1, 1]$.[7] A valid degree sequence must ensure $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$. In Section 7.1.2, we will provide a generative process for sampling valid in-degrees and out-degrees. Particularly, we can build upon the configuration model (Molloy and Reed 1995, Newman et al. 2001, Chung and Lu 2002) to construct such directed random networks with power-law degree distributions.

The following Theorem 8 gives an upper bound on the asymptotic value of price discrimination for such random networks with power-law degree distributions.

THEOREM 8. *For any exponent $\alpha > 2$, consider the sequence of power-law distributions with c.d.f. $F(n)$. For the sequence of random networks $G(n)$ with in- and out-degrees i.i.d. with distribution $F(n)$ and any degree correlation $\rho_{a,b} \in [-1, 1]$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = o(1)$.*

Moreover, for different values of α, we have the following bounds: the expected regret

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \begin{cases} O(n^{4-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2 < \alpha < 3 \\ O(n^{\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3 \\ O\left(n^{\frac{\alpha-2}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

[7] While we do not restrict the possible values of $\rho_{a,b}$, the correlation between in-degree and out-degree in such networks are usually much higher than -1 because of the highly asymmetric nature of the degree distribution and the fact that the sum of in-degrees of all nodes and the sum of out-degrees of all nodes are equal to the number of links in the network.

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for any $\epsilon > 0$; and the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \begin{cases} O(n^{3-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2 < \alpha < 3 \\ O(n^{-\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3 \\ O\left(n^{-\frac{1}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

for any $\epsilon > 0$.

Theorem 8 shows that, for random networks with power-law degree distributions, the expected regret grows sub-linearly in n, and the expected fractional regret vanishes asymptotically. This yields the same conclusion as in Theorem 2 for Erdös-Renyi networks, suggesting that the value of price discrimination is small for a broader class of random networks. Furthermore, the expected fractional regret decays at least polynomially in the size of the network. We also point out that, the degree correlation $\rho_{a,b}$ does not affect the rate of decay of the expected regret or the expected fractional regret; however, as we will see in the numerical experiments in Section 7.1.2, the degree correlation $\rho_{a,b}$ does play a role in determining the magnitude of the value of price discrimination.

We now provide a proof sketch for Theorem 8. To apply Theorem 7, we need to obtain a bound on the second moment of the degree distribution in multigraph $(G(n)+G(n)^T)$, and a lower bound on the maximum degree of $(G(n)+G(n)^T)$. In particular, the degree of node i in $(G(n)+G(n)^T)$ is the sum of the in-degree a_i and out-degree b_i for node i. Thus, we can calculate the second moment of the degree distribution based on the p.d.f. of the power-law distribution in equation (6.1) and the specified degree correlation $\rho_{a,b}$. Different values of α lead to different (exact) orders of the second moment in terms of n. To find the lower bound of the maximum degree in the multigraph $(G(n)+G(n)^T)$, we first obtain a bound on the maximum in-degree/out-degree in network $G(n)$, which is also a natural lower bound for the maximum degree in multigraph $(G(n)+G(n)^T)$. We then show that with high probability, the maximum in-degree or out-degree is lower bounded by $x_{min} n^{\frac{1-\delta}{\alpha-1}}$, for any small enough $\delta > 0$. This result is consistent with the structural cutoff result in Newman (2003) that provides the expected maximum degree in power-law networks. Finally, the (exact) rate of the second moment and the lower bound of the in-/out-degree jointly determine the upper bounds we obtain in Theorem 8. The detailed proof of Theorem 8 is provided in Appendix E.

To conclude this section, we provide an example of a sequence of random networks $G(n)$ where the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] \neq o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] \neq o(1)$. We consider a sequence of directed, binary star networks $G(n)$ with growing size n. In the nth network, one of the n consumers is randomly picked as the sink, and the remaining $n-1$ consumers are the sources. In other words, exactly one random consumer is being influenced by everyone else in

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network $G(n)$. The optimal pricing strategy should charge the sink consumer the highest price, and charge everyone else in the network the same (lower) price. It can be verified that for $G(n)$, the spectral norm $\|G(n)+G(n)^T\|=\sqrt{n-1}$, the expected profit under optimal uniform pricing

$$\mathbf{E}_G[\pi_0(n)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2(n+2\rho\sqrt{n-1}),$$

and the expected optimal profit

$$\mathbf{E}_G[\pi^*(n)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2\frac{n+2\rho\sqrt{n-1}}{1-\rho^2}.$$

So the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2\frac{\rho^2}{1-\rho^2}(n+2\rho\sqrt{n-1})=\Theta(n)$, and the expected fraction regret $\mathbf{E}_G[R_F(\mathbf{p}_0)]=\rho^2=\Theta(1)$.

7. Numerical Experiments

In this section, we conduct numerical experiments to validate our theoretical results. Our numerical experiments consist of four parts. In the first part, we show how the value of price discrimination changes as the network size n increases, under different network densities in Erdös-Renyi networks, and under different exponents and degree correlations in power-law networks. In the second part, we show how $p(n)$ affects the value of price discrimination, under a given network size n, in Erdös-Renyi networks. In the third part, we test a variant of our model to demonstrate the robustness of our results. Finally, we investigate the value of price discrimination on some real-world networks.

7.1. Impact of Network Size on the Value of Price Discrimination

In our first set of experiments, we investigate the impact of network size on the value of price discrimination, for both Erdös-Renyi networks and power-law networks.

7.1.1. Erdös-Renyi Networks

For Erdös-Renyi networks, we numerically study the decreasing rates of average fractional regret for different $p(n)$ as n increases. We consider $\rho\in\{0.6,0.7,0.8,0.9\}$. For each $n\in\{500,1000,2000,5000,10000,20000,50000,100000\}$, we randomly generate $N=100$ number of independent directed Erdös-Renyi networks with $p(n)\in\{\frac{1}{n^{1.1}},\frac{1}{n},\frac{(\log n)^2}{n}\}$. For each realization of $G(n,p(n))$, we compute the corresponding fractional regret according to equation (3.6). We take average of the fractional regrets under the same combination of n and $p(n)$ and obtain the average fractional regret $\overline{R_F}(\mathbf{p}_0)$ corresponding to that combination.

To better compare the decreasing rates of the average fractional regret for different $p(n)$, we present the simulation results as scatter plots in log-log scale in Figure 3. The scatter plots for different values of $p(n)$ appear to follow straight lines in the log-log scale for the values of n of our

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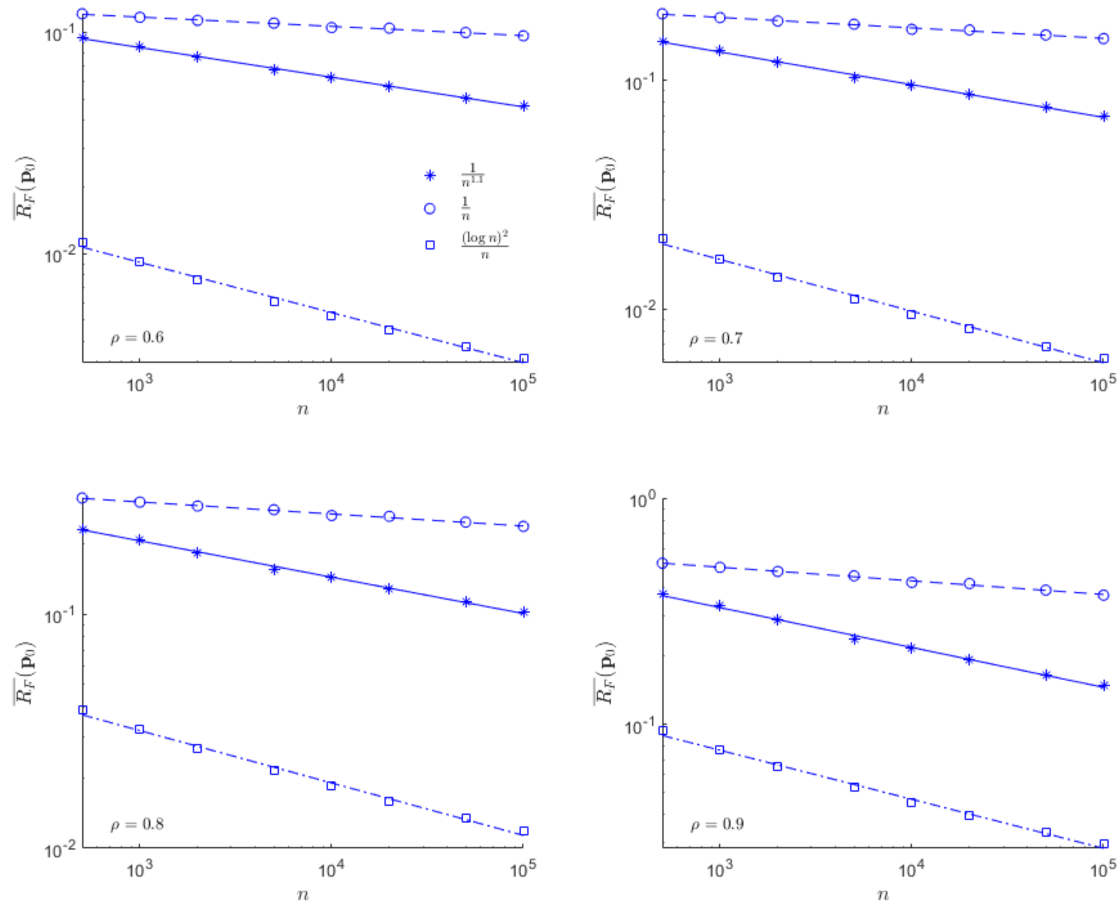
Figure 3 **Log-log plot of average fractional regrets under uniform pricing across different n in Erdös-Renyi networks.**



choice, but they are not straight lines essentially. For comparison, we add lines that are the best-fit regression lines in the log-log scale. The slopes of lines in the log-log scale plot reflect the decreasing rates under different circumstances. From Figure 3, we know that all lines have negative slopes. This implies that in general, the average fractional regret decays as n increases, with different decay rates for different ranges of $p(n)$. This is consistent with our result in Theorem 2. In particular, the results show that when $p(n)$ is relatively small or relatively large, the slope is steeper and the decreasing rate is faster. When $p(n)$ is moderately large, the slope is flatter and the decay rate is slower. This is consistent with results in Section 5. As ρ increases, the value of price discrimination increases uniformly across various $p(n)$. However, we can see from Figure 3 that, the decay rates for different $p(n)$ are roughly the same under different values of ρ. This implies that the decay rates of the expected fractional regret does not depend on the choices of ρ.

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7.1.2. Power-Law Networks

For random networks with power-law degree distributions, we numerically study the decreasing rates of the average fractional regret for different power-law exponents and different pairwise correlations between the in-degree and out-degree of nodes as n increases.

We first provide an approach to construct a valid correlated in-degree sequence $a_1, ..., a_n$ and out-degree sequence $b_1, ..., b_n$. We first sample i.i.d. in-degrees $a_1, ..., a_n$ from the power-law distribution F. Without loss of generality, we assume the sequence $a_1, ..., a_n$ is sorted in descending order. Next, we sample i.i.d. random variables $Z_1, ..., Z_n$ as follows: For each i, $Z_i = 1$ with probability $|\theta|$ and $Z_i = 0$ with probability $1 - |\theta|$, where $\theta \in [-1, 1]$ is a parameter we use to control the correlation between in-degrees and out-degrees (it is not necessarily the correlation $\rho_{a,b}$ between the in- and out-degree sequence). We define sets of nodes $I_0 = \{i : Z_i = 0, 1 \le i \le n\}$, and $I_1 = \{i : Z_i = 1, 1 \le i \le n\}$. The out-degrees $b_1, ..., b_n$ are constructed by a permutation of $a_1, ..., a_n$ as follows. If $\theta \ge 0$, then we set $b_i = a_i$ for $i \in I_1$, and set $\{b_i : i \in I_0\}$ by a random permutation of $\{a_i : i \in I_0\}$. If $\theta < 0$, then we set $b_i = a_{n-i+1}$ and set $\{b_i : i \in I_1\}$ by a random permutation of $\{a_{n-i+1} : i \in I_1\}$. By this construction, both the in-degrees and the out-degrees follow the power-law distribution F and their sum of in-degrees and out-degrees are equal. When $\theta > 0$, the in- and out-degree sequences have a positive correlation; when $\theta < 0$, the two degree sequences have a negative correlation.

Moreover, when $\theta \ge 0$, the correlation between in-degree and out-degree is $\theta + O(n^{-1})$.[8] So asymptotically the correlation is $\rho_{a,b} = \theta$. This is because when n is large (as in our simulation), an approximately equivalent representation of b_i is $b_i = Z_i a_i + (1 - Z_i)X$, where X follows the same power-law distribution F and is independent of a_i. We can then verify that $\mathbf{Var}(b_i) = \mathbf{Var}(a_i)$ and $\mathbf{Cov}(a_i, b_i) = \theta \mathbf{Var}(a_i)$. Therefore, for large networks, asymptotically $\mathbf{corr}(a_i, b_i) = \theta$ for any i and $\rho_{a,b} = \theta$.

We now describe the details of our numerical experiments. We set $x_{min} = 2$ and $\rho = 0.8$ for all cases. For each $\alpha \in \{2.5, 2.75, 3.0, 3.25, 3.5\}$, we generate a sequence of networks with different sizes n according to the power-law distribution specified in equation (6.1) and (6.2). For each α and n, we generate the in-degree sequence and out-degree sequence with different values of the parameter $\theta \in \{-1, -0.5, 0, 0.5, 1\}$ according to the approach described earlier in this section. For the in- and out-degree sequences associated with each θ, we use configuration model (Molloy and Reed 1995, Newman et al. 2001) to construct the directed random network $G(n, \alpha, \theta)$ with the specified degrees, and compute the corresponding fractional regret according to equation (3.6). For

[8] Since there is a small probability $\frac{1}{|I_0|}$ (which in expectation is $\frac{1}{n(1-\theta)}$) that $b_i = a_i$ for any $i \in I_0$, therefore b_i and a_i have a small correlation, which vanishes as $n \to \infty$.

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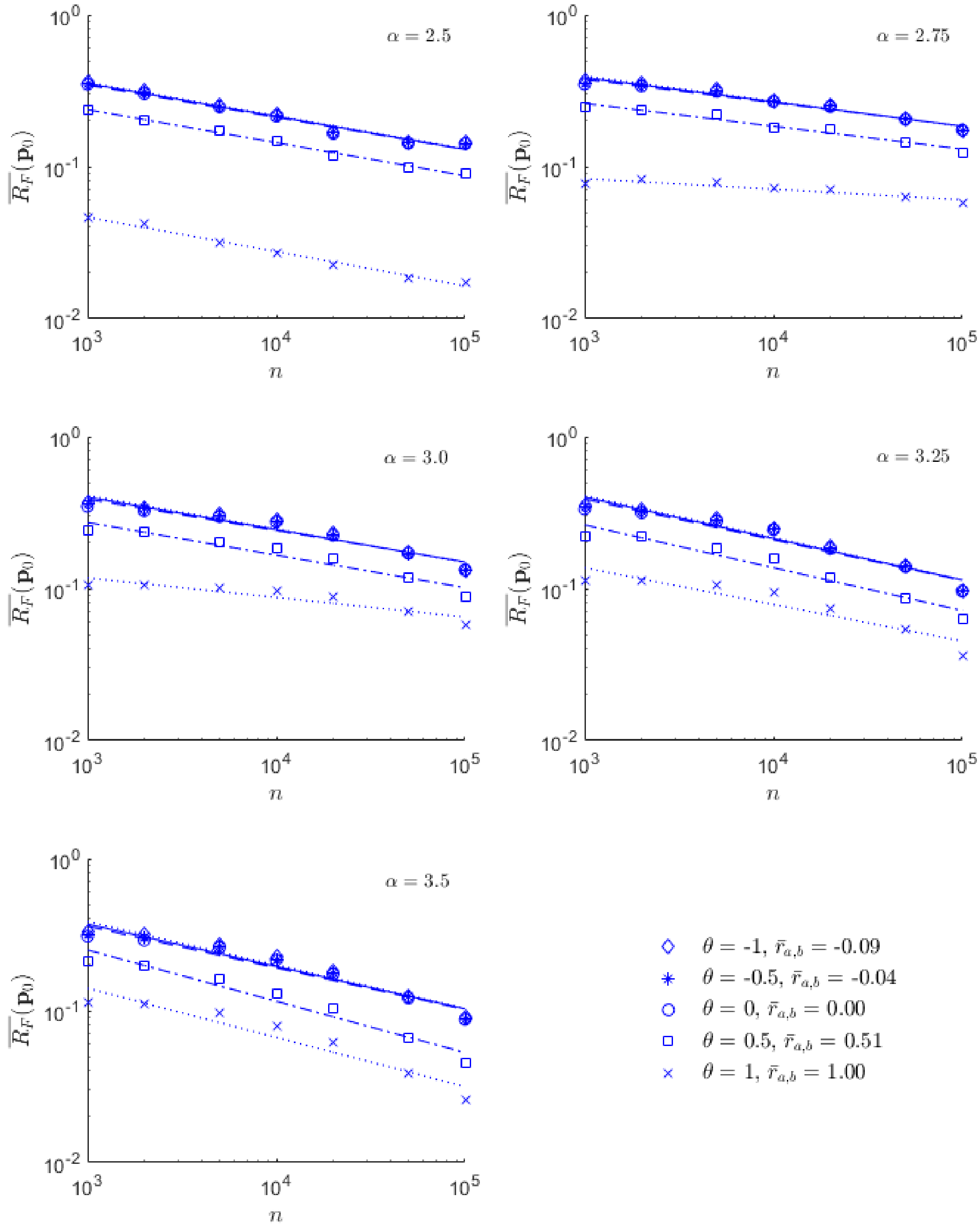
Figure 4 Log-log plot of average fractional regrets under uniform pricing across different n in power-law networks.



each combination of parameters, we repeat the above process for $N = 100$ times independently, and compute the average fractional regret $\overline{R_F}(\mathbf{p}_0)$.

We present our results as scatter plots in the log-log scale, as shown in Figure 4. We add the best-fit regression lines to better illustrate the decreasing trends of the fractional regret. We also

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list the average sample Pearson correlation between the in- and out-degree sequence (denoted as $\bar{r}_{a,b}$) corresponding to each θ. We observe that, for the simulated power-law networks, the average fractional regret decreases as the network size n increases. This observation is consistent with our theoretical results about the value of price discrimination on power-law networks in Theorem 8. Moreover, we observe that the slopes of the regression lines in Figure 4 are quite steep (as compared to the cases where $p(n) = \frac{1}{n}$ in Figure 3), and thus the decreasing rates of the average fractional regret in these power-law networks are relatively fast. This observation is again consistent with our theoretical results in Theorem 8, as we show that the rates of decreasing for power-law networks are at least as fast as polynomial decay.

Moreover, we observe that the average fractional regret generally increases as we decrease θ from 1 to 0, and further decreasing θ from 0 to -1 does not have much impact on the magnitude of the average fractional regret. On one hand, this observation suggests that a higher positive correlation leads to a smaller value of price discrimination. This can be explained by the level of imbalance in the network. As we increase the degree correlation to a higher positive value, the incoming and outgoing influence of each consumer in the network become more balanced, leading to a smaller value of price discrimination. On the other hand, when the correlation is zero or negative, the network is very unbalanced and thus demonstrates a relatively high value of price discrimination. When $\theta \leq 0$, the value of price discrimination remains similar across different values of θ. We point out that θ is not equivalent to the degree correlation $\rho_{a,b}$ when $\theta < 0$. Since the degrees of nodes are discrete and the degree distribution is highly asymmetric, a big mass of nodes would have in- or out-degrees concentrated around the lower bound of the support of the power-law distribution. Thus it is actually difficult to create power-law networks with significantly anti-correlated in- and out-degrees. Therefore, networks with different non-positive values of θ yield to similar degree sequences and hence similar value of price discrimination. We find that, even choosing $\theta = -1$ leads to an average sample correlation $\bar{r}_{a,b} = -0.09$. Therefore, the results corresponding to negative θ almost coincide with the results corresponding to $\theta = 0$.

7.2. Impact of Network Densities on the Value of Price Discrimination

In our second set of experiments, we consider the case when the underlying networks are directed Erdös-Renyi networks with size $n = 100,000$. In our numerical experiments, we choose $\rho \in \{0.6, 0.7, 0.8, 0.9\}$. For each ρ, we consider $p(n) \in \{\frac{1}{n^{1.3}}, \frac{1}{n^{1.1}}, \frac{1}{n}, \frac{\log\log n}{n}, \frac{\sqrt{\log n}}{n}, \frac{\log n}{n}, \frac{(\log n)^2}{n}\}$. Then for each $p(n)$, we randomly generate $N = 100$ independent directed Erdös-Renyi networks with 100,000 nodes. For each realization of $G(n, p(n))$, we compute the corresponding fractional regret according to equation (3.6).

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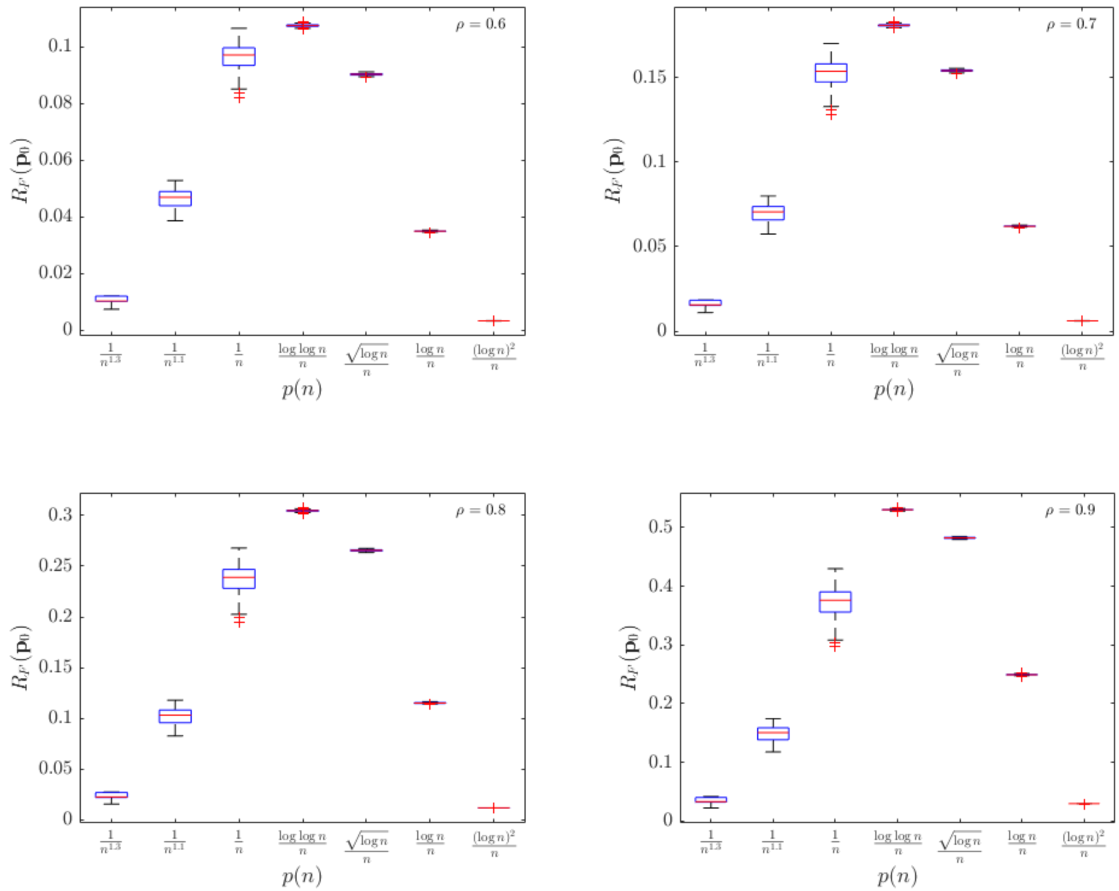
Figure 5 **Box-plot of fractional regrets under uniform pricing in Erdös-Renyi networks ($n = 100,000$).**



The values of fractional regrets are presented in the form of boxplots in Figure 5. We can see that for each ρ, the values of fractional regret first increase and then decrease as $p(n)$ increases. The peak is reached when $p(n) = \frac{\log\log n}{n}$. When $p(n)$ is very small ($p(n) = \frac{1}{n^{1.3}}$) or very large ($p(n) = \frac{(\log n)^2}{n}$), the fractional regret is less than 5%. In addition, as the value of ρ increases, the value of price discrimination increases uniformly across all values of $p(n)$. This implies that, larger ρ leads to larger value of price discrimination under the same n and $p(n)$.

Recall that in Theorems 3, 4, and 5, our theoretical bounds of expected fractional regret under different ranges of $p(n)$ also yield similar trends for a fixed network size: when the network density is relatively small or relatively large, the expected fraction regret decreases very fast and has a small value of price discrimination for large n; when the network density is moderately large, the expected fractional regret decreases slowly such that for decently large network size n, the value of price discrimination is non-negligible.

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7.3. The Value of Price Discrimination with Degree Normalization

Next, we investigate the robustness of our results. We perform similar analysis as in Section 7.1 and Section 7.2, but with a variant of our model. In particular, instead of normalizing a consumer's local network effect by the spectral norm of $(G + G^T)$, we normalize it by the total amount of influence (or the total in-degree of G) received by the consumer. Specifically, we consider the setting in which the preferences of the consumers are represented by the following utility function:

$$u_i(x_i, \mathbf{x}_{-i}, p_i) = ax_i - x_i^2 + x_i \sum_{j \neq i} \frac{G_{ij}}{\sum_{j \neq i} G_{ij}} x_j - p_i x_i.$$

For any i, if $\sum_{j \neq i} G_{ij} = 0$ then $G_{ij} = 0$ for all $j \neq i$, and we consider $0/0$ to be 0. This utility function implies that, the more total influence consumer i receives, the less important his/her single neighbor's purchasing decision would be to him/her. In this setting, the normalization factor could be different for different consumers.

In the simulation, we study the impact of network sizes and network densities on the value of price discrimination under this model variant for Erdös-Renyi networks. The simulation process to study the impact of network sizes (network densities, resp.) is exactly the same as that described in Section 7.1.1 (Section 7.2, resp.), except that we modify the normalization factor of each node as the total number of in-degrees of the node. The simulation results are presented in Figure 6. As we can see from Figure 6, the observed patterns/trends in this model variant are similar to our results in the base model. Particularly, the average fractional regret still decreases as we increase the network size n, and the decay rates for different $p(n)$ are different. For a given network size, the values of fractional regret also firstly increase and then decrease, as we increase the network density. The peak is reached when the network density is moderately large. This set of simulation results suggests that our results may be applicable to a larger set of models, and the theoretical results we obtained are robust with respect to this model variant.

7.4. The Value of Price Discrimination on Real Networks

We select five social network datasets, namely 'Epinions1', 'LiveJournal1', 'Slashdot0811', 'Slashdot0902', and 'Pokec', from Stanford Large Network Dataset Collection (Leskovec and Krevl 2014), and use the converted version of the data from The University of Florida sparse matrix collection (Davis and Hu 2011). All the selected datasets are directed binary social networks. A brief description of each network is summarized in Table 7.

For each dataset, we extract the adjacency matrix G and calculate the corresponding fractional regret according to equation (3.6) with given value of ρ. The basic statistics of the selected social

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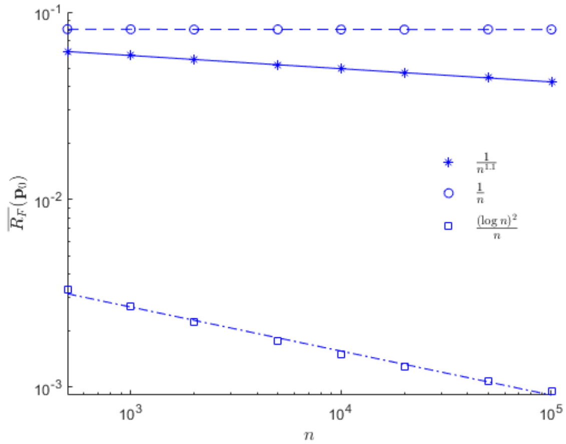
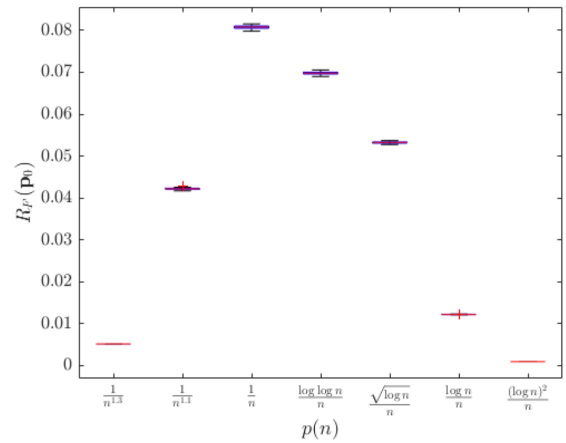
Figure 6 Fractional regrets under uniform pricing in Erdös-Renyi networks with degree normalization.



(a) Regret for different n and $p(n)$ in log-log scale.



(b) Regrets for different $p(n)$ ($n = 100,000$).

Table 7 Brief descriptions of selected real-world networks.

Name of Networks	Description
Epinions1	A who-trust-whom online social network of a a general consumer review site Epinions.com. Members of the site can decide whether to "trust" each other. All the trust relationships interact and form the Web of Trust which is then combined with review ratings to determine which reviews are shown to the users. (Richardson et al. 2003)
LiveJournal1	LiveJournal is a free on-line community with almost 10 million members, allowing members to maintain journals, individual and group blogs, and declare which other members are their friends. (Backstrom et al. 2006, Leskovec et al. 2009)
Slashdot0811	Slashdot is a technology-related news website known for its specific user community with a feature allowing users to tag each other as friends or foes. The network contains friend/foe links between the users. The links don't distinguish friend or foe relationship, so the network is binary and non-negative. The network was obtained in November 2008. (Leskovec et al. 2009)
Slashdot0902	Same as 'Slashdot0811'. The network was obtained in February 2009. (Leskovec et al. 2009)
Pokec	Pokec is the most popular online social network in Slovakia. The popularity of network has not changed even after the coming of Facebook. The network contains oriented user friendship data. (Takac and Zabovsky 2012)

networks and the fractional regrets under different values of ρ are summarized in Table 8. The degree correlation coefficient is the sample Pearson correlation coefficient of the in-degree and out-

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degree of the network. The level of symmetry is defined as the ratio between the number of links whose reverse is also in the network and the total number of links in the network.

Table 8 **Results from real-world data.**

Name of Networks	Epinions1	LiveJournal1	Slashdot0811	Slashdot0902	Pokec
Number of Nodes	75,888	4,847,571	77,360	82,168	1,632,803
Average In/Out Degree	6.7051	14.2326	11.7046	11.5430	18.7546
Degree Correlation Coefficient	0.5491	0.6490	0.9547	0.9343	0.7150
Level of Symmetry	41%	75%	87%	84%	54%
Fractional Regret ($\rho = 0.9$)	0.1627	0.0048	0.0265	0.0417	0.0423
Fractional Regret ($\rho = 0.8$)	0.0611	0.0029	0.0088	0.0131	0.0190
Fractional Regret ($\rho = 0.7$)	0.0289	0.0019	0.0040	0.0058	0.0109
Fractional Regret ($\rho = 0.6$)	0.0150	0.0012	0.0021	0.0030	0.0066

The results are presented in Table 8. In Table 8, we can see that the value of price discrimination is very small for the largest network 'LiveJournal', with the fractional regret of uniform pricing being less than 1% for all values of ρ. For the other networks with medium size, the value of price discrimination is generally small. However, it could already be significant in certain contexts, especially when ρ is closer to 1. Moreover, we note that in Table 8, for the networks of similar sizes ('Epinions1', 'Slashdot0811' and 'Slashdot0902'), usually the higher the level of symmetry of the network is, the smaller the fractional regret is. Overall, although these real-world networks do not necessarily follow the theoretical models we investigate, the results can provide part support for our main results.

8. Conclusions and Future Work

In this paper, we study the asymptotic value of price discrimination in large random social networks. We find that, for Erdös-Renyi random networks, the value of price discrimination is not a positive fraction of the profit asymptotically for all network densities. Yet, when the average degree of the network stays as a constant or grows slower than the logarithm of the size of the network, the rate of expected fractional regret decays slowly and therefore, for decently large networks there could still be non-negligible value of price discrimination. The results for very sparse networks (average degree decreasing in the size of the network) are driven by the fact that the network is fragmented enough for the influence of any individual to be small, while for dense networks (average degree

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increasing at least as the logarithm of the size of the network) the results are driven by relatively equal positions of individuals in the network.

We also extend our analysis to random networks with general degree distributions. We propose a general framework based on degree information to provide bounds for the expected value of price discrimination in random networks. We apply the general framework to random networks with power-law degree distributions, and show that the value of price discrimination (in terms of expected fractional regret) vanishes as the size of the network increases, for any exponent $\alpha > 2$.

Given our results, it would appear that the firms need to be more careful about using price discrimination, because the value of such discriminative pricing policies under many cases may not be substantial, while the inequity and the lack of transparency in pricing can lead to lower consumer satisfaction and mistrust. Moreover, our analysis and results can serve as a first step in addressing the value of other marketing strategies on social networks, such as product promotions or referral programs.

Our analysis is purely structural and ignores the heterogeneity in consumer preferences. While consumer preferences play a role in generating value from price discrimination, the interaction between heterogeneous consumer preferences and network structure and its role in generating value from price discrimination is an important direction of future work. Another possible future direction of work could focus on the moderate-sized networks where there may be some value of price discrimination. Network information is often noisy and using optimal pricing assuming perfect network information may be sub-optimal. The design of robust pricing policies under noisy network information can also be a future research direction.

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Appendix

In the following, we suppress the indexes and represent $G(n,p(n))$ or $G(n)$ as G and $p(n)$ as p for simplicity.

A. Proof of Theorem 1

Proof By definition, G is a price discrimination free network if and only if for any $\rho \in (0,1)$,

$$\mathbf{1}^T \left(I - \frac{\rho}{\|G+G^T\|} \left(G+G^T\right) \right)^{-1} \mathbf{1} = \mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|} G \right)^{-1} \mathbf{1}.$$

Or equivalently,

$$\sum_{k=0}^{\infty} \left(\frac{2\rho}{\|G+G^T\|} \right)^k \mathbf{1}^T \left(\frac{G+G^T}{2} \right)^k \mathbf{1} = \sum_{k=0}^{\infty} \left(\frac{2\rho}{\|G+G^T\|} \right)^k \mathbf{1}^T G^k \mathbf{1}.$$

Define $f(\rho) = \sum_{k=0}^{\infty} \left(\frac{2\rho}{\|G+G^T\|} \right)^k \mathbf{1}^T \left(\frac{G+G^T}{2} \right)^k \mathbf{1}$ and $g(\rho) = \sum_{k=0}^{\infty} \left(\frac{2\rho}{\|G+G^T\|} \right)^k \mathbf{1}^T G^k \mathbf{1}$ as two functions of ρ. In order for $f(\rho) = g(\rho)$ for all ρ, we need to have for all k,

$$\mathbf{1}^T \left(\frac{G+G^T}{2} \right)^k \mathbf{1} = \mathbf{1}^T G^k \mathbf{1}. \tag{A.1}$$

Conversely, if equation (A.1) holds for all positive integer k, then we can immediately tell that $R(\mathbf{p}_0) = 0$ for any ρ, and G is a price discrimination free network.

Now we show that (A.1) is equivalent to $G^k \mathbf{1} = (G^T)^k \mathbf{1}$ for all k. Note that equation (A.1) always holds for $k = 1$. When $k = 2$, expanding both sides of the equation yields that equation (A.1) holds if and only if $G\mathbf{1} = G^T \mathbf{1}$. Now assume that $G^k \mathbf{1} = \left(G^T\right)^k \mathbf{1}$ holds for any integer $k \leq K$. When $k = K+1$, expanding $\mathbf{1}^T \left(\frac{G+G^T}{2} \right)^{2K+2} \mathbf{1} = \mathbf{1}^T G^{2K+2} \mathbf{1}$ will lead to

$$\frac{1}{2} \mathbf{1}^T G^{K+1} \left(G^T\right)^{K+1} \mathbf{1} + \frac{1}{2} \mathbf{1}^T \left(G^T\right)^{K+1} G^{K+1} \mathbf{1} = \mathbf{1}^T G^{2K+2} \mathbf{1},$$

and this equation holds if and only if $G^{K+1} \mathbf{1} = \left(G^T\right)^{K+1} \mathbf{1}$.

Thus by induction, equation (A.1) holds for all positive integer k if and only if $G^k \mathbf{1} = \left(G^T\right)^k \mathbf{1}$ for all positive integer k. Thus we finish the proof of the theorem. □

B. A General Lower Bound on the Expected Regret

In this subsection, we provide a general lower bound for the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]$ by decomposing the network into components, and considering only components with size 2.

By equation (3.5), the regret is given by

$$R(\mathbf{p}_0) = \frac{1}{2} \left(\frac{a-c}{2} \right)^2 \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G+G^T\|} \left(G+G^T\right) \right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|} G \right)^{-1} \mathbf{1} \right]. \tag{B.1}$$

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If the network G is not connected, we can decompose the network G into components,[9] and by reorganizing the orders of nodes in the matrix representation, we obtain a new adjacency matrix G that is a block diagonal matrix.[10] In particular, we assume G has the following form

$$G = \begin{bmatrix} \mathbf{0} & & & & & \\ & G_1 & & & & \\ & & G_2 & & & \\ & & & G_3 & & \\ & & & & \ddots & \\ & & & & & G_L \end{bmatrix},$$

where $G_i,\ i = 1, 2, ..., L$ represents the adjacency matrix of each component, and L is the total number of components in G. Then $(G + G^T)$ has the following form:

$$G + G^T = \begin{bmatrix} \mathbf{0} & & & & & \\ & G_1 + G_1^T & & & & \\ & & G_2 + G_2^T & & & \\ & & & G_3 + G_3^T & & \\ & & & & \ddots & \\ & & & & & G_L + G_L^T \end{bmatrix}.$$

We can rewrite equation (B.1) as

$$R(\mathbf{p}_0) = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{i=1}^{L} \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G + G^T\|}\left(G_i + G_i^T\right)\right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G + G^T\|} G_i\right)^{-1} \mathbf{1}\right],$$

where $\frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G+G^T\|}\left(G_i + G_i^T\right)\right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G+G^T\|} G_i\right)^{-1} \mathbf{1}\right] \geq 0$, following from the optimality of discriminative pricing. In particular, if $G_{sub} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ represents the adjacency matrix of a component with 2 nodes and 1 link, and there are l number of such components in the network G. Then only considering such components gives a lower bound of the regret,

$$R(\mathbf{p}_0) \geq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{i=1}^{l} \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G + G^T\|}\left(G_{sub} + G_{sub}^T\right)\right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G + G^T\|} G_{sub}\right)^{-1} \mathbf{1}\right].$$

The regret of setting uniform pricing on G_{sub} is given by

$$\begin{aligned} R_{G_{sub}}(\mathbf{p}_0) &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left[\mathbf{1}^T \left(I - \frac{\rho}{\|G + G^T\|}\left(G_{sub} + G_{sub}^T\right)\right)^{-1} \mathbf{1} - \mathbf{1}^T \left(I - \frac{2\rho}{\|G + G^T\|} G_{sub}\right)^{-1} \mathbf{1}\right] \\ &= \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{\|G + G^T\|\left(\|G + G^T\| - \rho\right)}. \end{aligned}$$

[9] The network G is a directed network. Here we consider the component as a subgraph of G such that any two vertices in the subgraph are connected to each other by paths if we ignore the directions of the network.

[10] We call it G again since the two adjacency matrices are equivalent.

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The expected number of components with adjacency matrix G_{sub} in the Erdös-Renyi network G is $\mathbf{E}(l) = 2\binom{n}{2}p(1-p)^{4(n-2)+1}$. Therefore, the lower bound of expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]$ is

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \left(\frac{a-c}{2}\right)^2 n(n-1)p(1-p)^{4(n-2)+1}\mathbf{E}_G\left[\frac{\rho^2}{\|G+G^T\|\left(\|G+G^T\|-\rho\right)}\right]. \tag{B.2}$$

C. Proofs of Structural Properties of Erdös-Renyi Networks

C.1. Proof of Lemma 1

Proof When $p = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p = \omega(n^{-2})$, we must have $0 < \epsilon < 1$. Then there exist some real number $m > 1$ such that $\frac{1}{m} \leq \epsilon$, and $p = O\left(n^{-\frac{m+1}{m}}\right)$. In the following, we prove the results for $p = O\left(n^{-\frac{m+1}{m}}\right)$ for some $m > 1$ (not necessary an integer) and $p = \omega(n^{-2})$.

In random graph theory, for a given property M, *threshold function* $t(n)$ is defined as a function that satisfies

$$P(G \text{ has the property } M) \to 0, \text{ if } \frac{p}{t(n)} \to 0,$$

and

$$P(G \text{ has the property } M) \to 1, \text{ if } \frac{p}{t(n)} \to \infty.$$

From Corollary 1 of Erdos and Rényi (1960), when m is an integer, $t(n) = n^{-\frac{m+2}{m+1}}$ is a threshold function for the emergence of a tree with $m+2$ nodes in the network G. Thus when $p = o(n^{-\frac{m+2}{m+1}})$ for some integer m,

$$\lim_{n\to\infty} P\left(G \text{ has at least one tree with } m+2 \text{ nodes}\right) = 0.$$

This implies that

$$\lim_{n\to\infty} P\left(G \text{ has at least one star with } m+2 \text{ nodes}\right) = 0,$$

and

$$\lim_{n\to\infty} P\left(G \text{ has at least one star with equal to or more than } m+2 \text{ nodes}\right) = 0.$$

When $p = O\left(n^{-\frac{m+1}{m}}\right)$ for some $m > 1$, we know that $p = o(n^{-\frac{\lfloor m+2\rfloor}{\lfloor m+1\rfloor}})$, where $\lfloor x\rfloor$ represents the largest integer that is less than or equal to x. Therefore,

$$\lim_{n\to\infty} P(\Delta \leq \lfloor m+2\rfloor) = 1.$$

Since $p = \omega(n^{-2})$ and $t(n) = n^{-2}$ is a threshold function for the emergence of an edge (Erdos and Rényi 1960), therefore,

$$\lim_{n\to\infty} P\left(G \text{ has at least one edge}\right) = 1,$$

and

$$\lim_{n\to\infty} P(\Delta \geq 1) = 1.$$

For any undirected network G, $\sqrt{\Delta} \leq \|G\| \leq \Delta$. Therefore when G is an undirected Erdös-Renyi network with the specified network density, asymptotically almost surely $1 \leq \|G\| \leq \lfloor m+2\rfloor$. Recall that m is some real number such that $m > 1$ and $\frac{1}{m} \leq \epsilon$, if we define $m(\epsilon) = \lfloor m+2\rfloor$, then we prove that asymptotically almost surely $1 \leq \|G\| \leq m(\epsilon)$, where $m(\epsilon)$ is a constant that only depends on ϵ. □

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C.2. Proof of Lemma 2

Proof If a network G (undirected) has degree distribution $d \sim P(d)$ and the expected degree of a node is $\mathbf{E}(d)$, then the degree distribution of a neighboring node (a node reached following an edge) is given by $\tilde{P}(d) = \frac{dP(d)}{\mathbf{E}(d)}$ (Newman et al. 2001). This is because there are $n\mathbf{E}(d)$ end points among all the edges in the network, and there are d stubs at every node of degree d. Following an edge, the edge chooses its end point randomly, and the probability that it ends at a specific node of degree d is $\frac{d}{n\mathbf{E}(d)}$ (for large enough n). Since there are $nP(d)$ nodes of degree d in the network, the probability that, following an edge which randomly picks its end point, we reach a neighboring node of degree d is $nP(d)\frac{d}{n\mathbf{E}(d)}$, or equivalently, $\tilde{P}(d)$. Therefore, the expected degree of the neighboring node is $\tilde{\mathbf{E}}(d) = \frac{\mathbf{E}(d^2)}{\mathbf{E}(d)} = \frac{\mathbf{Var}(d)+(\mathbf{E}(d))^2}{\mathbf{E}(d)}$.

In the multigraph $(G+G^T)$, where G is a directed Erdös-Renyi network with probability p, we can view the in/out-degree distribution as i.i.d. binomial distribution $d \sim Bin(2(n-1), p)$ as n grows large enough. Thus $\mathbf{E}(d) = 2(n-1)p$, $\mathbf{Var}(d) = 2(n-1)p(1-p)$, $\tilde{\mathbf{E}}(d) = 1-3p+2np$, where d denotes the in/out-degree in the multigraph $(G+G^T)$.

The number of walks of length 2 starting from a random node v is given by $\sum_{u \in N^+(v)} deg^+(u)$, where $N^+(v)$ is the set of out-going neighbors of v and $deg^+(u)$ is the out-degree of node u. Thus the expected number of walks of length 2 starting from v is $\mathbf{E}\left[\sum_{u \in N^+(v)} deg^+(u)\right] = \mathbf{E}(d)\tilde{\mathbf{E}}(d)$. Since we choose v randomly, the expected number of walks of length 2 in the multigraph $G+G^T$ is

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right] = n\mathbf{E}(d)\tilde{\mathbf{E}}(d) = 2n(n-1)p(1-3p+2np).$$

This finishes our proof. □

C.3. Proof of Lemma 3

Proof $(G+G^T)$ is a symmetric matrix, and $\|\left(G+G^T\right)^k\| = \|G+G^T\|^k$ for any integer $k>0$. By definition, the largest eigenvalue of a symmetric matrix B is defined as $\|B\| = \max_\xi \frac{\xi^T B\xi}{\xi^T\xi}$. Therefore, for $k \geq t$,

$$\frac{\mathbf{1}^T\left(G+G^T\right)^{\frac{t}{2}}\left(G+G^T\right)^{k-t}\left(G+G^T\right)^{\frac{t}{2}}\mathbf{1}}{\mathbf{1}^T\left(G+G^T\right)^{\frac{t}{2}}\left(G+G^T\right)^{\frac{t}{2}}\mathbf{1}} \leq \|\left(G+G^T\right)^{k-t}\| = \|G+G^T\|^{k-t}.$$

This is equivalent to

$$\mathbf{1}^T\left(G+G^T\right)^k\mathbf{1} \leq \|G+G^T\|^{k-t}\mathbf{1}^T\left(G+G^T\right)^t\mathbf{1}.$$

Thus we finish the proof. □

C.4. Proof of Lemma 4

Proof Suppose $G^k = ((G^k)_{ij})$. Then

$$(G^k)_{ij} = \sum\nolimits_{l(1),l(2),\ldots,l(k-1)} G_{il(1)}G_{l(1)l(2)}\ldots G_{l(k-1)j}.$$

Since there might be repeated arcs in $\{il(1), l(1)l(2), \ldots, l(k-1)j\}$, therefore

$$\mathbf{E}_G((G_k)_{ij}) \geq \sum\nolimits_{l(1),l(2),\ldots,l(k-1)} p^k.$$

Now we consider all possible combinations of $\{l(1), l(2), \ldots, l(k-1)\}$ that can result in walks of length k in G. We can choose $l(1) \neq i$, $l(2) \neq l(1)$, ..., $l(k-1) \neq l(k-2)$ and $l(k-1) \neq j$. Thus there are $(n-1)^{k-2}(n-2)$ choices. This gives $\mathbf{E}_G((G_k)_{ij}) \geq (n-1)^{k-2}(n-2)p^k$. Since there are n^2 elements in the matrix G, $\mathbf{E}_G(\mathbf{1}^T G^k\mathbf{1}) \geq n^2(n-1)^{k-2}(n-2)p^k$. □

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C.5. Proof of Proposition 1

Proof We first prove part (i). To characterize the difference, we first develop a method to estimate the number of walks of length k in multigraph $(G+G^T)$ and network G.

For each integer $k \geq 0$ and integer $2 \leq l \leq \min\{k+1, n\}$, we define the set of motifs inducing walks of length k with l distinct nodes. The set of such motifs is represented as:

$$\begin{aligned} F_k^l = & \{(i_0, i_1, ..., i_k) \mid i_0, i_1, ..., i_k \in L,\ i_0 = 0,\ i_1 \neq i_0, i_2 \neq i_1, ..., i_k \neq i_{k-1}, \\ & i_h \leq 1 + \max_{h' \leq h}\{i_{h'}\},\ \text{and}\ \max_{0 \leq h \leq k}\{i_h\} = l-1\}, \end{aligned}$$

where $L = \{0, 1, ..., l-1\}$ is the set of dummy nodes. The set of all motifs inducing walks of length k is

$$F_k = \cup_{2 \leq l \leq \min\{k+1,n\}} F_k^l.$$

For any motif $f \in F_k$, $l(f)$ denotes the number of nodes in the motif.

Any walk of length k in a directed multigraph is represented as $w = (v_0, e_1, v_1, ..., e_k, v_k)$, where $v_0, v_1, ..., v_k$ are the nodes and $e_1, ..., e_k$ are the links such that e_h is a link between the ordered pair of nodes (v_{h-1}, v_h).

In the network G, on one hand, for any walk $w = (v_0, e_1, v_1, ..., e_k, v_k)$, there exists a unique motif $f = (i_0, i_1, ..., i_k) \in F_k$, and a unique injection function $g_{w,f} : \{0, 1, ..., l(f)-1\} \to N$ that maps the indices of nodes in the motif to the actual set of nodes, denoted as N. In particular, $v_0 = g_{w,f}(i_0), ..., v_k = g_{w,f}(i_k)$ and the link e_h, $1 \leq h \leq k$, is the unique link between the ordered pair of nodes (v_{h-1}, v_h). On the other hand, given any motif $f = (i_0, i_1, ..., i_k) \in F_k$ and any injection function $g_{w,f} : \{0, 1, ..., l(f)-1\} \to N$, we can find a unique walk $w = (v_0, e_1, v_1, ..., e_k, v_k)$ of length k in the network G, such that $v_0 = g_{w,f}(i_0), ..., v_k = g_{w,f}(i_k)$ and the link e_h, $1 \leq h \leq k$, is the unique link between the ordered pair of nodes (v_{h-1}, v_h). For any $f_i \in F_k$, let $h_i(G)$ denote the number of possible walks associated to motif f_i in the network G. The above argument implies that $h_i(G)$ is equal to the number of such injection functions.

Similarly, in the multigraph $(G+G^T)$, on one hand, for any walk $w = (v_0, e_1, v_1, ..., e_k, v_k)$, there exists a unique motif $f = (i_0, i_1, ..., i_k) \in F_k$, a unique injection function $g_{w,f} : \{0, 1, ..., l(f)-1\} \to N$, and a unique binary vector $\mathbf{e}_w \in \{0,1\}^k$, such that $v_0 = g_{w,f}(i_0), ..., v_k = g_{w,f}(i_k)$ and the link e_h, $1 \leq h \leq k$, is the unique link between the ordered pair of nodes (v_{h-1}, v_h) in G if $e_w(h) = 1$, and is the unique link between the ordered pair of nodes (v_{h-1}, v_h) in G^T if $e_w(h) = 0$. On the other hand, given any motif $f = (i_0, i_1, ..., i_k) \in F_k$, any injection function $g_{w,f} : \{0, 1, ..., l(f)-1\} \to N$ and any binary vector $\mathbf{e}_w \in \{0,1\}^k$, we can find a unique walk $w = (v_0, e_1, v_1, ..., e_k, v_k)$ of length k in the multigraph $(G+G^T)$, such that $v_0 = g_{w,f}(i_0), ..., v_k = g_{w,f}(i_k)$ and the link e_h, $1 \leq h \leq k$, is the unique link between the ordered pair of nodes (v_{h-1}, v_h) in G if $e_w(h) = 1$, and is the unique link between the ordered pair of nodes (v_{h-1}, v_h) in G^T if $e_w(h) = 0$. For any $f_i \in F_k$, let $h_i(G+G^T)$ denote the number of possible walks associated to motif f_i in the multigraph $(G+G^T)$. The above argument implies that $h_i(G+G^T)$ is equal to 2^k multiplied by the number of such injection functions.

Moreover, by definition we can see that for any motif $f_i \in F_k$,

$$h_i(G+G^T) = 2^k h_i(G).$$

In particular, if $l \leq n$ and $f_i \in F_k^l$, then $h_i(G) = P(n,l)$, and $h_i(G+G^T) = 2^k P(n,l)$. Here, $P(n,l)$ means the $l-$permutation of n, i.e. $P(n,l) = \frac{n!}{(n-l)!}$.

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In this way, we have established the relationship between motifs of walks of length k, and the possible walks of length k in network G or multigraph $(G+G^T)$. If we consider all distinct motifs of walks of length k, and sum up the number of walks associated with each motif, we obtain the count of walks of length k in the network.

Let m_k denote the number of distinct motifs of walks of length k, i.e. $m_k = |F_k|$. Note that any motif of walks of length k contains at most $k+1$ nodes. There is only one motif that contains $k+1$ nodes, i.e. the motif representing the paths of length k (without repeated nodes), and all other motifs contain at most k nodes. The upper bound of number of possible walks (without self-loops) build on k nodes starting from a given node is $(k-1)^k$. This is also an upper bound of possible number of distinct motifs with at most k nodes in the network. Therefore,

$$m_k \leq 1 + (k-1)^k.$$

To count the number of walks, we define binary random variables representing the existence of walks in the network G as follows,

$$X_j^i(G) = \begin{cases} 1, & \text{if walk } j \text{ associated to motif } f_i \text{ of walks in network } G \text{ exists} \\ 0, & \text{else,} \end{cases}$$

with $i = 1, ..., m_k$ and $j = 1, ..., h_i(G)$. In particular, if f_i is a motif consisting s distinct ordered pairs of nodes, or equivalently, s distinct links, then

$$P(X_j^i(G) = 1) = p^s, \tag{C.1}$$

for $j = 1, 2, ..., h_i(G)$.

Similarly, we define binary random variables representing the existence of walks in the multigraph $(G+G^T)$ as follows,

$$X_j^i(G+G^T) = \begin{cases} 1, & \text{if walk } j \text{ associated to motif } f_i \text{ of walks in multigraph } G+G^T \text{ exists} \\ 0, & \text{else,} \end{cases}$$

with $i = 1, ..., m_k$ and $j = 1, ..., h_i(G+G^T)$. In particular, if f_i is a motif consisting s distinct ordered pairs of nodes, or equivalently, s distinct links, then

$$P(X_j^i(G+G^T) = 1) \geq p^s, \tag{C.2}$$

for $j = 1, 2, ..., h_i(G+G^T)$. This is because with probability p, a link and its reverse appear simultaneously in the multigraph $(G+G^T)$, i.e. the number of links between the ordered pair of nodes (u, v) is equal to the number of links between the ordered pair of nodes (v, u). This correlation of links increases the probability of the existence of a given walk.

Now we can represent the number of walks of length k in the network through the summation of binary random variables as follows,

$$\mathbf{1}^T G^k \mathbf{1} = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G)} X_j^i(G),$$

and

$$\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G+G^T)} X_j^i(G+G^T).$$

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Taking expectation on both sides of the expressions, we obtain

$$\mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right] = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G)} \mathbf{E}_G[X_j^i(G)] = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G)} P(X_j^i(G) = 1), \tag{C.3}$$

and

$$\mathbf{E}_G\left[\mathbf{1}^T \left(G + G^T\right)^k \mathbf{1}\right] = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G+G^T)} \mathbf{E}_G[X_j^i(G+G^T)] = \sum_{i=1}^{m_k} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T) = 1). \tag{C.4}$$

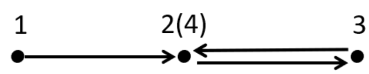
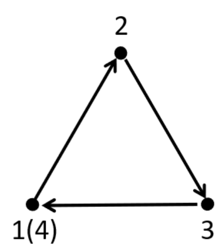
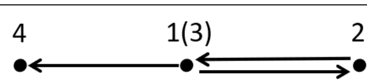
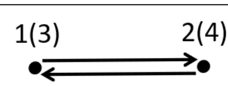
From equation (C.1), (C.2), (C.3) and (C.4), and the relationship that $h_i(G+G^T) = 2^k h_i(G)$, we can conclude that

$$\mathbf{E}_G\left[\mathbf{1}^T \left(G + G^T\right)^k \mathbf{1}\right] \geq 2^k \mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right].$$

This is valid for any integer $k \geq 0$. Therefore we finish the proof of part (i).

We illustrate our proof ideas with an example where $k = 3$ and $n \geq 4$, and summarize the results in Table 9. For walks with small lengths, we are able to characterize the exact expression of the expected number of walks of given length in the network G and multigraph $(G + G^T)$. The labels of nodes in the motifs represent the order of visiting a node in the walk.

Table 9 Expected number of walks of length 3 in the network G and multigraph $(G+G^T)$.

	Motif	$\mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right]$	$\mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1}\right]$
1		$P(n,4)p^3$	$P(n,4)\,(2^3p^3)$
2		$P(n,3)p^3$	$P(n,3)\,(2^2p^2 + 2^2p^3)$
3		$P(n,3)p^3$	$P(n,3)\,(2^3p^3)$
4		$P(n,3)p^3$	$P(n,3)\,(2^2p^2 + 2^2p^3)$
5		$P(n,2)p^2$	$P(n,2)\,(2p + 6p^2)$

Now we prove part (ii). We decompose the set of all distinct motifs inducing walks of length k, F_k, into three subsets:

- The first subset is denoted as A_1, where $A_1 \subseteq F_k$ and for any $f \in A_1$, for any $a, b \in \{0, 1, ..., l(f) - 1\}$, if a is immediately followed by b in the sequence of nodes in the motif f, then b is never immediately followed by a in the sequence. For example, motif 1 and 3 in Table 9.

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- The second subset is denoted as A_2, where $A_2 = F_k^k \setminus A_1$, and F_k^k is the set of motifs inducing walks of length k with k distinct nodes. For example, motif 2 and 4 in Table 9.
- The third subset is denoted as A_3, where $A_3 = F_k \setminus (A_1 \cup A_2)$. For example, motif 5 in Table 9.

From definition of the three subsets, we know that they are distinct subsets of F_k, and the union $A_1 \cup A_2 \cup A_3 = F_k$. Therefore, we can rewrite equations (C.3) and (C.4) as

$$\begin{aligned}&\mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right]\\ &= \sum_{f_i \in A_1} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1) + \sum_{f_i \in A_2} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1) + \sum_{f_i \in A_3} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1),\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1}\right]\\ &= \sum_{f_i \in A_1} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) + \sum_{f_i \in A_2} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1)\\ &+ \sum_{f_i \in A_3} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1).\end{aligned}$$

We can analyze the difference between $2^k \mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right]$ and $\mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1}\right]$ for each subset individually.

We first consider the subset A_1. For any motif $f_i \in A_1$, by definition of A_1, the probability that a walk associated with f_i exists is p^k in the network G or $(G+G^T)$, i.e. $P(X_j^i(G)=1) = p^k$, for any $j = 1, 2, ..., h_i(G)$, and $P(X_j^i(G+G^T)=1) = p^k$, for any $j = 1, 2, ..., h_i(G+G^T)$. Since $h_i(G+G^T) = 2^k h_i(G)$, we know the following relationship:

$$\sum_{f_i \in A_1} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) - 2^k \sum_{f_i \in A_1} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1) = 0. \tag{C.5}$$

This implies that walks associated with motifs in A_1 do not contribute to the difference between $\mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1}\right]$ and $2^k \mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right]$.

We then consider the subset A_2. From the definition of A_2, we know that there are exactly $(k-1)$ number of distinct motifs in A_2, i.e. $|A_2| = k-1$. For any motif $f_i \in A_2$, since $f_i \in F_k^k$, we know that $h_i(G) = P(n,k)$, and $h_i(G+G^T) = 2^k P(n,k)$. Since motifs in A_2 consist of k distinct nodes, one ordered pair of nodes and its reverse pair present exactly once in any motif in A_2. We denote such pairs of nodes as (u,v) and (v,u). The probability that a walk associated with $f_i \in A_2$ exists is p^k in the network G, i.e. $P(X_j^i(G)=1) = p^k$, for any $j = 1, 2, ..., h_i(G)$. However, the probability that a walk associated with $f_i \in A_2$ exists in the multigraph $(G+G^T)$ is either p^k or p^{k-1}. Consider the pair of nodes u and v such that both (u,v) and (v,u) appear in f_i, and focus on the link between (u,v) and the link between (v,u). For a walk $X_j^i(G+G^T)$ associated with motif $f_i \in A_2$, two cases might happen: (1) if both links between the pair of nodes are from G or both links are from G^T, then $P(X_j^i(G+G^T)=1) = p^{k-1}$; (2) if one of the links is from G and the other one is from G^T, then $P(X_j^i(G+G^T)=1) = p^k$. We can also see that half of the walks associated with a motif $f_i \in A_2$ follow case (1), while the other half of the walks follow case (2). Therefore, we know that

$$\sum_{f_i \in A_2} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1) = (k-1)P(n,k)p^k,$$

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and

$$\sum_{f_i \in A_2} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) = (k-1)P(n,k)\left(2^{k-1}p^{k-1} + 2^{k-1}p^k\right).$$

This implies that

$$\begin{aligned}&\sum_{f_i \in A_2} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) - 2^k \sum_{f_i \in A_2} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1)\\ =&(k-1)2^{k-1}P(n,k)(1-p)p^{k-1}\\ \leq&(k-1)2^{k-1}P(n,k)p^{k-1}.\end{aligned} \tag{C.6}$$

We finally consider the subset A_3. Motifs in A_3 have at most $k-1$ nodes. We only consider the multigraph $(G+G^T)$ in the following discussion. For a walk with l nodes, we need at least $l-1$ links with distinct pairs of end points to ensure the walk is connected. Therefore, the probability of the existence of a walk with l nodes is at most p^{l-1}. Thus for any motif $f_i \in F_k^l$,

$$\sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) \leq \sum_{j=1}^{h_i(G+G^T)} p^{l-1} = 2^k P(n,l)p^{l-1}.$$

Since $0 \leq k < n$ and $p \geq \frac{1}{n-k+1}$, we know that for any $2 \leq l \leq k-1$, $(n-l+1)p \geq (n-k+1)p \geq 1$, and therefore $2^k P(n,l)p^{l-1}$ increases in l. Since $l \leq k-1$ for motifs in A_3, we know that for any motif $f_i \in A_3$,

$$\sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) \leq 2^k P(n,k-1)p^{k-2}.$$

The number of distinct motifs in A_3 is at most $(k-1)^k$, since $m_k \leq 1+(k-1)^k$ and both A_1 and A_2 are non-empty subsets. Therefore,

$$\sum_{f_i \in A_3} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) \leq (k-1)^k 2^k P(n,k-1)p^{k-2}.$$

Since $2^k \sum_{f_i \in A_3} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1)$ is non-negative, this also implies the bound

$$\sum_{f_i \in A_3} \sum_{j=1}^{h_i(G+G^T)} P(X_j^i(G+G^T)=1) - 2^k \sum_{f_i \in A_3} \sum_{j=1}^{h_i(G)} P(X_j^i(G)=1) \leq (k-1)^k 2^k P(n,k-1)p^{k-2}. \tag{C.7}$$

Combine results from equation (C.5), (C.6) and (C.7), we conclude that when $0 \leq k < n$ and $\frac{1}{n-k+1} \leq p < 1$,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^k\mathbf{1}\right] - 2^k\mathbf{E}_G\left[\mathbf{1}^T G^k \mathbf{1}\right] \leq (k-1)2^{k-1}P(n,k)p^{k-1} + (k-1)^k 2^k P(n,k-1)p^{k-2}.$$

This finishes the proof of part (ii). □

C.6. Proof of Proposition 2

Proof Our proof is based on the following version of Chernoff bounds: Let $X_1, ..., X_n$ be n independent Bernoulli random variables where $P(X_i=1)=p_i$, and let $X=\sum_{i=1}^n X_i$ and $\mu=\sum_{i=1}^n p_i$, then for any $0<\delta<1$, $P(X \leq (1-\delta)\mu) \leq e^{-\frac{\mu\delta^2}{2}}$, and $P(X \geq (1+\delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$.

For Erdös-Renyi network G, the in/out degree of a node can be viewed as the sum of $(n-1)$ i.i.d. Bernoulli random variables. For simplicity of the proof, we can approximate it as the sum of n i.i.d. Bernoulli random

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variables, and this will not impact the proceeding of the proof. For any fixed node v, applying the Chernoff bound gives

$$P(deg^{+/-}(v) \le (1-\delta(n))c(n)\log n) \le e^{-\frac{\delta(n)^2 c(n)\log n}{2}},$$

$$P(deg^{+/-}(v) \ge (1+\delta(n))c(n)\log n) \le e^{-\frac{\delta(n)^2 c(n)\log n}{3}}.$$

By union bound, the probability that any node in G has in/out degree less than $(1-\delta(n))c(n)\log n$ is at most $ne^{-\frac{\delta(n)^2 c(n)\log n}{2}}$, and the probability that any node in G has in/out degree greater than $(1+\delta(n))c(n)\log n$ is at most $ne^{-\frac{\delta(n)^2 c(n)\log n}{3}}$. Since $c(n)$ is a function of n such that $\lim_{n\to\infty} c(n) = +\infty$ and $c(n)\log n < n$ for all n, and $\delta(n)$ is a function of n such that $\delta(n) = \Theta(\frac{1}{\sqrt{c(n)}})$ and $\sqrt{\frac{12}{c(n)}} < \delta(n) < 1$, therefore $\lim_{n\to\infty} ne^{-\frac{\delta(n)^2 c(n)\log n}{2}} = 0$, and

$$\sum_{n=1}^{\infty} P(deg^{+/-}(v) \le (1-\delta(n))c(n)\log n, \forall v) \le \sum_{n=1}^{\infty} ne^{-\frac{\delta(n)^2 c(n)\log n}{2}} < \infty.$$

Similarly, $\lim_{n\to\infty} ne^{-\frac{\delta(n)^2 c(n)\log n}{3}} = 0$, and

$$\sum_{n=1}^{\infty} P(deg^{+/-}(v) \ge (1+\delta(n))c(n)\log n, \forall v) \le \sum_{n=1}^{\infty} ne^{-\frac{\delta(n)^2 c(n)\log n}{3}} < \infty.$$

Thus by Borel-Cantelli Lemma, we know that asymptotically almost surely, every node has in/out degree within the range of $[(1-\delta(n))c(n)\log n, (1+\delta(n))c(n)\log n]$. □

D. Proofs of Main Results for Erdös-Renyi Random Networks

Before proving the main theorems in Section 5.1, we first review the expression of expected regret, and show some general bounds of the expected regret that are useful in the proofs.

By equation (3.5), the expected regret is given by

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\mathbf{1}^T\left(I - \frac{\rho}{\|G+G^T\|}(G+G^T)\right)^{-1}\mathbf{1} - \mathbf{1}^T\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1}\right]. \tag{D.1}$$

The geometric series expansion of matrix gives the equivalent equation

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=0}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T(G+G^T)^k\mathbf{1} - \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}\right]. \tag{D.2}$$

Since $\mathbf{1}^T(G+G^T)^k\mathbf{1} = \mathbf{1}^T(2G)^k\mathbf{1}$ for $k = 0, 1$, therefore

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=2}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T(G+G^T)^k\mathbf{1} - \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}\right]. \tag{D.3}$$

By Lemma 3, we obtain an upper bound for the expected regret as

$$\mathbf{E}_G[R(\mathbf{p}_0)] \le \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=2}^{\infty} \mathbf{E}_G\left[\rho^k \left(\frac{1}{\|G+G^T\|}\right)^2 \mathbf{1}^T(G+G^T)^2\mathbf{1}\right]. \tag{D.4}$$

By definition $\rho \in (0,1)$, thus the geometric series converges with respect to k. Therefore,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \le \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\frac{\rho^2}{1-\rho}\left(\frac{1}{\|G+G^T\|}\right)^2 \mathbf{1}^T(G+G^T)^2\mathbf{1}\right]. \tag{D.5}$$

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D.1. Proof of Theorem 2

Proof Given any sequence p, we will have a sequence of Erdös-Renyi random networks G. Corresponding to the sequence of networks $(G+G^T)$, there is a sequence of spectral norms $\|G+G^T\|$.

In the following proof, we first show that under two different assumptions of the sequence of spectral norms, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$. We then argue that for any sequence of probabilities p, we can decompose the sequence into two subsequences, such that each subsequence satisfies one of the two assumptions. Therefore, we can derive the bounds of expected regrets for the two subsequences individually, and combine the two bounds to obtain a general upper bound of the expected regret.

Case 1. We first assume that $\|G+G^T\| = \omega(np)$ almost surely. From equation (D.5) we obtain an upper bound of the expected regret. Since G is non-empty, therefore $\|G+G^T\| \geq 1$.[11]

We now consider two possible sub-cases. When $np = o(1)$, since $\|G+G^T\| \geq 1$, therefore from equation (D.5),

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

By Lemma 2,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho} n(n-1)p(1-3p+2np).$$

Since we assume that $np = o(1)$, therefore

$$\mathbf{E}_G[R(\mathbf{p}_0)] = o(n).$$

When $np = \Omega(1)$, from equation (D.5) and the assumption that $\|G+G^T\| = \omega(np)$ almost surely,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\omega(np)}\right)^2 \mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

By Lemma 2,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\omega(np)}\right)^2 n(n-1)p(1-3p+2np).$$

Since we assume that $np = \Omega(1)$, therefore

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\omega(np)}\right)^2 \Theta(n^3p^2).$$

This implies that

$$\mathbf{E}_G[R(\mathbf{p}_0)] = o(n).$$

Case 2. We next assume that $\|G+G^T\| = O(np)$ almost surely. By equation (5.2.2), $\|G+G^T\| \geq (2+o(1))np$ almost surely. This implies that $\|G+G^T\| = \Theta(np)$ almost surely. We claim that in this case, $np = \omega(1)$, or equivalently $p = \omega(n^{-1})$. Otherwise if there is a subsequence $\{n'\} \subseteq \{n\}$ such that $n'p(n') = O(1)$, then from the asymptotic spectral results in Section 5.2.2, we know that $\|G(n')+G(n')^T\| = (2+o(1))\sqrt{\Delta} = \omega(n'p(n'))$ almost surely, which contradicts with the assumption that $\|G+G^T\| = O(np)$ almost surely.

[11] Consider a network G with n nodes and only one link, for example, G is a $n\times n$ matrix with 1 at the ijth element for a given pair (i,j) and 0 elsewhere. Then $(G+G^T)$ is a matrix with 1 at the ijth and jith element and 0 elsewhere. Let e_i denote a n-dimension unit vector that has 1 in the ith position and 0 everywhere else. By definition, $\|G+G^T\| = \max_\xi \frac{\xi^T(G+G^T)\xi}{\xi^T\xi} \geq \frac{(e_i+e_j)^T(G+G^T)(e_i+e_j)}{(e_i+e_j)^T(e_i+e_j)} = 1$. Therefore a non-empty network (with at lease one link) has spectral norm greater than or equal to 1.

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We decompose the geometric series in equation (D.3) into two parts,

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left(\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] + \mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))]\right),$$

where $\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))]$ and $\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))]$ denote the two parts respectively. In particular,

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] = \sum_{k=2}^{K(n,p)} \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} - \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}\right], \tag{D.6}$$

and

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] = \sum_{k=K(n,p)+1}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} - \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}\right]. \tag{D.7}$$

We choose $K(n,p) = \left\lceil \sqrt{\log(np)} \right\rceil$, where $\lceil x \rceil$ represents the smallest integer that is greater than or equal to x. The choice of $K(n,p)$ implies that $K(n,p) = o(n)$.

We first consider $\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))]$. From Proposition 1 part (i), for any integer $k \geq 0$,

$$\mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} - 2^k \mathbf{1}^T G^k \mathbf{1}\right] \geq 0.$$

Since $\|G+G^T\| \geq (2+o(1))np$ almost surely, from equation (D.6), $\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))]$ is upper bounded by

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] \leq \sum_{k=2}^{K(n,p)} \left(\frac{\rho}{2np}\right)^k \mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} - 2^k \mathbf{1}^T G^k \mathbf{1}\right].$$

Noticing that $2 \leq k \leq K(n,p) = o(n)$, and $p = \omega(n^{-1})$, from Proposition 1 part (ii),

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] \leq \sum_{k=2}^{K(n,p)} \left(\frac{\rho}{2np}\right)^k \left[(k-1)2^{k-1}P(n,k)p^{k-1} + (k-1)^k 2^k P(n,k-1)p^{k-2}\right].$$

Since $P(n,k) \leq n^k$, therefore

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] \leq \frac{1}{2p} \sum_{k=2}^{K(n,p)} (k-1)\rho^k + \frac{1}{np^2} K(n,p)^{K(n,p)} \sum_{k=2}^{K(n,p)} \rho^k.$$

Summing up the geometric series gives

$$\begin{aligned}&\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] \\ \leq& \frac{1}{2p(1-\rho)^2}\left[\left(K(n,p)-1\right)\rho^{K(n,p)+2} - K(n,p)\rho^{K(n,p)+1} + \rho^2\right] + \frac{\rho^2}{np^2(1-\rho)} K(n,p)^{K(n,p)}\left[1-\rho^{K(n,p)}\right].\end{aligned} \tag{D.8}$$

Since $K(n,p) = \left\lceil \sqrt{\log(np)} \right\rceil$, therefore $\log\left(K(n,p)^{K(n,p)}\right) = \left\lceil \sqrt{\log(np)} \right\rceil \log \left\lceil \sqrt{\log(np)} \right\rceil = o(\log(np))$. This implies that $K(n,p)^{K(n,p)} = o(np)$. Also because $np = \omega(1)$, $K(n,p) = \omega(1)$. Noticing that $\rho \in (0,1)$, therefore $\rho^{K(n,p)} = o(1)$. Also since $np = \omega(1)$, therefore $p^{-1} = o(n)$. This implies that

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] = O(p^{-1}) = o(n). \tag{D.9}$$

Now we consider $\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))]$. By Lemma 3, equation (D.7) is upper bounded by

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] \leq \sum_{k=K(n,p)+1}^{\infty} \mathbf{E}_G\left[\rho^k \left(\frac{1}{\|G+G^T\|}\right)^2 \mathbf{1}^T \left(G+G^T\right)^2 \mathbf{1}\right]. \tag{D.10}$$

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Since $\|G+G^T\| = \Theta(np)$ almost surely, we obtain

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] \leq \sum_{k=K(n,p)+1}^{\infty} \rho^k \left(\frac{1}{\Theta(np)}\right)^2 \mathbf{E}_G\left[\mathbf{1}^T \left(G+G^T\right)^2 \mathbf{1}\right].$$

Using Lemma 2 and computing the sum of geometric series,

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] \leq \frac{2\rho^{K(n,p)+1}}{1-\rho}\left(\frac{1}{\Theta(np)}\right)^2 n(n-1)p(1-3p+2np).$$

Since $K(n,p) = \omega(1)$ and $\rho \in (0,1)$, therefore $\rho^{K(n,p)+1} = o(1)$. Also in this case $np = \omega(1)$. This implies that

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] = o(n). \tag{D.11}$$

Combining equation (D.9) and (D.11),

$$\mathbf{E}_G[R(\mathbf{p}_0)] = o(n).$$

In this way, we show that the expected regret grows at the rate $o(n)$ under assumption $\|G+G^T\| = \omega(np)$ almost surely or assumption $\|G+G^T\| = O(np)$ almost surely.

We now consider a general probability sequence $p(n)$, and the sequence of spectral norms $\|G(n)+G(n)^T\|$ induced by $p(n)$. We call a subsequence $\{n'\} \subseteq \{n\}$ a maximal subsequence associated with $\omega(np(n))$ if, we can not find any subsequence $\{m'\} \subseteq \{n\} \setminus \{n'\}$ such that $\|G(m')+G(m')^T\| = \omega(m'p(m'))$ almost surely. Similarly, we call a subsequence $\{n'\} \subseteq \{n\}$ a maximal subsequence associated with $O(np(n))$ if, we can not find any subsequence $\{m'\} \subseteq \{n\} \setminus \{n'\}$ such that $\|G(m')+G(m')^T\| = O(m'p(m'))$ almost surely. We point that such choices of maximal subsequences are not unique.

For the sequence of spectral norms with indexes $\{n\}$, we find two subsequences with indexes $\{n_1\}$ and $\{n_2\}$, where $\{n_1\}$ is a maximal subsequence associated with $\omega(np(n))$, and $\{n_2\}$ is a maximal subsequence associated with $O(np(n))$. Note that one of the two subsequences can be empty, but they cannot both be empty subsequences. Assume that there exists a subsequence $\{m\} \subseteq \{n\} \setminus (\{n_1\} \cup \{n_2\})$. If $\|G(m) + G(m)^T\| = \omega(mp(m))$ almost surely, then this violates the definition of $\{n_1\}$. If $\|G(m)+G(m)^T\| \neq \omega(mp(m))$ almost surely, then there exists a subsequence $\{m_1\} \subseteq \{m\}$ such that $\|G(m_1) + G(m_1)^T\| = O(m_1 p(m_1))$ almost surely, and this violates the definition of $\{n_2\}$. Therefore, there does not exist any subsequence $\{m\}$ such that $\{m\} \subseteq \{n\} \setminus (\{n_1\} \cup \{n_2\})$. In other words, there are finitely many elements in $\{n\} \setminus (\{n_1\} \cup \{n_2\})$. Without loss of generality, we can assume that $\{n\} = \{n_1\} \cup \{n_2\}$ and this will not influence our limiting results.

The subsequence $\{n_1\}$ satisfies the condition that $\|G(n_1) + G(n_1)^T\| = \omega(n_1 p(n_1))$ almost surely and therefore within the subsequence $\{n_1\}$, $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n_1)$. The subsequence $\{n_2\}$ satisfies the condition that $\|G(n_2)+G(n_2)^T\| = O(n_2 p(n_2))$ almost surely and therefore within the subsequence $\{n_2\}$, $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n_2)$. The combination of the two subsequence traverses the whole sequence of networks. This implies that the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$ for any sequence of probabilities $p(n)$.

From equation (5.7), we know that $\pi^* = \Theta(n)$. Therefore, the expected fraction regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \mathbf{E}_G\left[\frac{R(\mathbf{p}_0)}{\pi^*}\right] = \frac{\mathbf{E}_G[R(\mathbf{p}_0)]}{\Theta(n)} = o(1).$$

□

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D.2. Proof of Theorem 3

Proof For the first part of the theorem, when $p = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p = \omega(n^{-2})$, from equation (5.6) we know that $\|G+G^T\| = \Theta(1)$ almost surely. This is because in this range of network densities, the network is very fragmented and the size of components are small, limiting the maximum degree of the network. From equation (D.5) we obtain an upper bound of the expected regret,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\Theta(1)}\right)^2 \mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

By Lemma 2,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\Theta(1)}\right)^2 n(n-1)p(1-3p+2np). \tag{D.12}$$

Since $p = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$, therefore equation (D.12) can be written as

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\frac{1}{\Theta(1)}\right)^2 \Theta(n^2p).$$

This implies the upper bound $\mathbf{E}_G[R(\mathbf{p}_0)] = O(n^2p)$.

We now provide the matching lower bound on the expected regret. Since $\|G+G^T\| = \Theta(1)$ almost surely, therefore from equation (B.2), we obtain

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{\Theta(1)\left(\Theta(1)-\rho\right)} n(n-1)p(1-p)^{4(n-2)+1}.$$

Since $p = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$, therefore

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2\Theta\left(n^2p\right)}{\Theta(1)\left(\Theta(1)-\rho\right)}.$$

This implies that $\mathbf{E}_G[R(\mathbf{p}_0)] = \Omega(n^2p)$. Combining the upper and lower bounds gives the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta(n^2p)$.

We next prove the rate for expected fractional regret. From equation (5.7), we know that $\pi^* = \Theta(n)$. Thus the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \mathbf{E}_G\left[\frac{R(\mathbf{p}_0)}{\pi^*}\right] = \frac{\mathbf{E}_G[R(\mathbf{p}_0)]}{\Theta(n)} = \Theta(np).$$

Thus we finish the proof the theorem. □

D.3. Proof of Theorem 4

Proof Consider the case when $p = \Theta(n^{-1})$. From equation (5.6) we know that $\|G+G^T\| = \Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right)$ almost surely. From equation (D.5) we obtain an upper bound of the expected regret,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\Theta\left(\sqrt{\frac{\log\log n}{\log n}}\right)\right)^2 \mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

By Lemma 2,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\left(\Theta\left(\sqrt{\frac{\log\log n}{\log n}}\right)\right)^2 n(n-1)p(1-3p+2np). \tag{D.13}$$

Since $p = \Theta(n^{-1})$, therefore

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\Theta\left(\frac{\log\log n}{\log n}n\right).$$

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This also implies the upper bound $\mathbf{E}_G[R(\mathbf{p}_0)] = O\left(\frac{\log\log n}{\log n} n\right)$.

Equation (B.2) holds as the lower bound of $\mathbf{E}_G[R(\mathbf{p}_0)]$. Without loss of generality we assume $p = \frac{w}{n}$ for some constant $w > 0$. Since $p = \frac{w}{n}$, and $\|G + G^T\| = \Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right)$ almost surely, therefore, from equation (B.2),

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \left(\frac{a-c}{2}\right)^2 \frac{w\rho^2(n-1)}{\Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right)\left(\Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right) - \rho\right)} \left(1 - \frac{w}{n}\right)^{4(n-2)+1}.$$

Simple calculation yields

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \left(\frac{a-c}{2}\right)^2 \frac{w\rho^2 \Theta(e^{-4w} n)}{\Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right)\left(\Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right) - \rho\right)}.$$

Therefore, $\mathbf{E}_G[R(\mathbf{p}_0)] = \Omega\left(\frac{\log\log n}{\log n} n\right)$. Combining both bounds, we obtain the bound $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(\frac{\log\log n}{\log n} n\right)$.

From equation (5.7), the optimal profit $\pi^* = \Theta(n)$. Thus we can compute the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \mathbf{E}_G\left[\frac{R(\mathbf{p}_0)}{\pi^*}\right] = \frac{\mathbf{E}_G[R(\mathbf{p}_0)]}{\Theta(n)} = \Theta\left(\frac{\log\log n}{\log n}\right).$$

□

D.4. Proof of Theorem 5

Proof In accordance with Proposition 2, we assume $p = \frac{c(n)\log n}{n}$, where $c(n)$ is a function of n such that $\lim_{n\to\infty} c(n) = +\infty$ and $c(n)\log n < n$ for all n. Actually every sequence of p in this range can be represented in this way.

From equation (5.6), when $p = \frac{c(n)\log n}{n}$, $\|G + G^T\| = (2 + o(1))np = \Theta(np)$ almost surely. This satisfies the condition in Case 2 in Section D.1, the proof of Theorem 2. Following the proof idea in Theorem 2, we can decompose the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]$ into two parts, as specified in equation (D.6) and (D.7). For $p = \frac{c(n)\log n}{n}$, we choose $K(n,p) = \left\lceil \sqrt{\log(np)} \right\rceil = \left\lceil \sqrt{\log(c(n)\log n)} \right\rceil$, where $\lceil x \rceil$ represents the smallest integer that is greater than or equal to x.

Since $K(n,p)^{K(n,p)} = o(np)$, $K(n,p) = \omega(1)$ and $\rho^{K(n,p)} = o(1)$, therefore the upper bound of $\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))]$ given by equation (D.9) still holds. Thus we can further write equation (D.9) as

$$\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))] = O(p^{-1}) = O\left(\frac{n}{c(n)\log n}\right).$$

We next provide a tighter upper bound for $\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))]$, which is given by equation (D.7). We decompose $G\mathbf{1} = c(n)\log n\mathbf{1} + \zeta_1$, and $G^T\mathbf{1} = c(n)\log n\mathbf{1} + \zeta_2$, where ζ_1 and ζ_2 denote the differences between the in/out degree and average in/out degree of the random network G. This also implies that $(G + G^T)\mathbf{1} = 2c(n)\log n\mathbf{1} + \zeta_1 + \zeta_2$. The discounted sum of walks of different lengths on the network $(G + G^T)$ is given by

$$\begin{aligned}
&\sum_{k=K(n,p)+1}^{\infty} \left(\frac{\rho}{\|G + G^T\|}\right)^k \mathbf{1}^T \left(G + G^T\right)^k \mathbf{1} \\
&= \left(\frac{\rho}{\|G + G^T\|}\right)^{K(n,p)+1} \mathbf{1}^T \left(G + G^T\right)^{K(n,p)+1} \mathbf{1} \\
&\quad + \sum_{k=K(n,p)+1}^{\infty} \left(\frac{\rho}{\|G + G^T\|}\right)^k \mathbf{1}^T \left(G + G^T\right)^k \left[\frac{2\rho c(n)\log n}{\|G + G^T\|}\mathbf{1} + \frac{\rho}{\|G + G^T\|}(\zeta_1 + \zeta_2)\right].
\end{aligned}$$

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By Proposition 2, every element in vector ζ_1 and ζ_2 is upper bounded by $\delta(n)c(n)\log n$ asymptotically almost surely, where $\delta(n) = \Theta(\frac{1}{\sqrt{c(n)}})$. Therefore, asymptotically almost surely,

$$\begin{aligned}
&\sum_{k=K(n,p)+1}^{\infty} \left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T \left(G+G^T\right)^k \mathbf{1} \\
&\leq \left(\frac{\rho}{\|G+G^T\|}\right)^{K(n,p)+1} \mathbf{1}^T \left(G+G^T\right)^{K(n,p)+1} \mathbf{1} \\
&\quad + \left(\frac{2\rho c(n)\log n}{\|G+G^T\|} + \frac{2\rho\delta(n)c(n)\log n}{\|G+G^T\|}\right) \sum_{k=K(n,p)+1}^{\infty} \left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T \left(G+G^T\right)^k \mathbf{1}.
\end{aligned} \tag{D.14}$$

Similarly, we can provide a lower bound to $\sum_{k=K(n,p)+1}^{\infty} \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}$. By Proposition 2, every element in vector ζ_1 is lower bounded by $-\delta(n)c(n)\log n$ asymptotically almost surely. Therefore, asymptotically almost surely,

$$\begin{aligned}
&\sum_{k=K(n,p)+1}^{\infty} \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1} \\
&\geq \left(\frac{2\rho}{\|G+G^T\|}\right)^{K(n,p)+1} \mathbf{1}^T G^{K(n,p)+1} \mathbf{1} \\
&\quad + \left(\frac{2\rho c(n)\log n}{\|G+G^T\|} - \frac{2\rho\delta(n)c(n)\log n}{\|G+G^T\|}\right) \sum_{k=K(n,p)+1}^{\infty} \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}.
\end{aligned} \tag{D.15}$$

For simplicity, we denote

$$S_1 = \left(\frac{\rho}{\|G+G^T\|}\right)^{K(n,p)+1} \mathbf{1}^T \left(G+G^T\right)^{K(n,p)+1} \mathbf{1},$$

$$S_2 = \left(\frac{2\rho}{\|G+G^T\|}\right)^{K(n,p)+1} \mathbf{1}^T G^{K(n,p)+1} \mathbf{1},$$

$$A = \sum_{k=K(n,p)+1}^{\infty} \left(\frac{\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T \left(G+G^T\right)^k \mathbf{1},$$

and

$$B = \sum_{k=K(n,p)+1}^{\infty} \left(\frac{2\rho}{\|G+G^T\|}\right)^k \mathbf{1}^T G^k \mathbf{1}.$$

We subtract equation (D.15) from (D.14), and obtain the following difference:

$$A - B \leq S_1 - S_2 + (1-\delta(n)+2\delta(n))\frac{2\rho c(n)\log n}{\|G+G^T\|} A - (1-\delta(n))\frac{2\rho c(n)\log n}{\|G+G^T\|} B, \ a.a.s.$$

Rearranging the terms,

$$A - B \leq \frac{S_1 - S_2 + 2\delta(n)\frac{2\rho c(n)\log n}{\|G+G^T\|} A}{1-(1-\delta(n))\frac{2\rho c(n)\log n}{\|G+G^T\|}}, \ a.a.s.$$

Since $\|G+G^T\| = (2+o(1))np = \Theta(np)$ almost surely, therefore taking expectations on both sides,

$$\mathbf{E}_G[A-B] \leq \frac{\mathbf{E}_G[S_1-S_2] + 2\delta(n)\frac{2\rho c(n)\log n}{\Theta(np)}\mathbf{E}_G(A)}{1-(1-\delta(n))\frac{2\rho c(n)\log n}{\Theta(np)}}.$$

Equivalently,

$$\mathbf{E}_G[A-B] \leq \frac{\mathbf{E}_G[S_1-S_2] + \Theta(1)\delta(n)\mathbf{E}_G(A)}{\Theta(1)}. \tag{D.16}$$

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By the definition of S_1, S_2,

$$\mathbf{E}_G[S_1 - S_2] = \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^{K(n,p)+1}\left(\mathbf{1}^T\left(G+G^T\right)^{K(n,p)+1}\mathbf{1} - 2^{K(n,p)+1}\mathbf{1}^T G^{K(n,p)+1}\mathbf{1}\right)\right].$$

From part (i) of Proposition 1, $\mathbf{E}_G[S_1 - S_2] \geq 0$. Since $\|G+G^T\| \geq (2+o(1))np$ almost surely, therefore

$$\mathbf{E}_G[S_1 - S_2] \leq \left(\frac{\rho}{2np}\right)^{K(n,p)+1}\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^{K(n,p)+1}\mathbf{1} - 2^{K(n,p)+1}\mathbf{1}^T G^{K(n,p)+1}\mathbf{1}\right].$$

From part (ii) of Proposition 1, we obtain an upper bound

$$\begin{aligned}\mathbf{E}_G[S_1 - S_2] \leq \left(\frac{\rho}{2np}\right)^{K(n,p)+1} & \left[K(n,p)2^{K(n,p)}P(n,K(n,p)+1)p^{K(n,p)}\right. \\ & \left. + K(n,p)^{K(n,p)+1}2^{K(n,p)+1}P(n,K(n,p))p^{K(n,p)-1}\right].\end{aligned}$$

Since $P(n,k) \leq n^k$, therefore

$$\mathbf{E}_G[S_1 - S_2] \leq \frac{K(n,p)}{2p}\rho^{K(n,p)+1} + \frac{K(n,p)^{K(n,p)+1}}{np^2}\rho^{K(n,p)+1}.$$

Because $K(n,p)^{K(n,p)} = o(np)$ and $K(n,p) = \omega(1)$,

$$\mathbf{E}_G[S_1 - S_2] = o(p^{-1}) = o\left(\frac{n}{c(n)\log n}\right). \tag{D.17}$$

By Lemma 3, $\mathbf{E}_G(A)$ is upper bounded by

$$\mathbf{E}_G(A) \leq \sum_{k=K(n,p)+1}^{\infty}\mathbf{E}_G\left[\rho^k\left(\frac{1}{\|G+G^T\|}\right)^2\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

Since $\|G+G^T\| = \Theta(np)$ almost surely, therefore

$$\mathbf{E}_G(A) \leq \sum_{k=K(n,p)+1}^{\infty}\rho^k\left(\frac{1}{\Theta(np)}\right)^2\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

Using Lemma 2 and computing the sum of geometric series,

$$\mathbf{E}_G(A) \leq \frac{2\rho^{K(n,p)+1}}{1-\rho}\left(\frac{1}{\Theta(np)}\right)^2 n(n-1)p(1-3p+2np).$$

Therefore,

$$\delta(n)\mathbf{E}_G(A) = O\left(n\delta(n)\rho^{K(n,p)}\right).$$

Now we compare the rate of $\delta(n)\rho^{K(n,p)}$ and $\frac{1}{c(n)\log n}$. By Proposition 2, $\delta(n) = \Theta(\frac{1}{\sqrt{c(n)}})$. Also $K(n,p)$ can be approximated by $\sqrt{\log(c(n)\log n)}$. Since

$$\log\log(\sqrt{c(n)}\log n) = o\left(\log\log(c(n)\log n)\right),$$

therefore

$$\log(\sqrt{c(n)}\log n) = o\left(-(\log\rho)\sqrt{\log(c(n)\log n)}\right),$$

where $\rho \in (0,1)$ is a constant as defined before. This further implies that

$$\sqrt{c(n)}\log n = o\left(\rho^{-\sqrt{\log(c(n)\log n)}}\right).$$

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Therefore,

$$\rho^{\sqrt{\log(c(n)\log n)}} = o\left(\frac{1}{\sqrt{c(n)}\log n}\right),$$

and

$$\delta(n)\rho^{K(n,p)} = o\left(\frac{1}{c(n)\log n}\right). \tag{D.18}$$

Applying equation (D.18),

$$\delta(n)\mathbf{E}_G(A) = o\left(\frac{n}{c(n)\log n}\right). \tag{D.19}$$

From equation (D.16), (D.17) and (D.19),

$$\mathbf{E}_G\left[A - B\right] = o\left(\frac{n}{c(n)\log n}\right).$$

This is equivalent to

$$\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))] = o\left(\frac{n}{c(n)\log n}\right).$$

Combining the upper bound for $\mathbf{E}_G[R_1(\mathbf{p}_0, K(n,p))]$ and $\mathbf{E}_G[R_2(\mathbf{p}_0, K(n,p))]$, we conclude that

$$\mathbf{E}_G\left[R(\mathbf{p}_0)\right] = O\left(\frac{n}{c(n)\log n}\right).$$

Now we provide a matching lower bound for $\mathbf{E}_G\left[R(\mathbf{p}_0)\right]$. From equation (D.3), we get the expression for $\mathbf{E}_G\left[R(\mathbf{p}_0)\right]$. If we only evaluate the geometric series at $k=2$, then we obtain a lower bound

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\left(\frac{\rho}{\|G+G^T\|}\right)^2 \mathbf{1}^T\left(G+G^T\right)^2\mathbf{1} - \left(\frac{2\rho}{\|G+G^T\|}\right)^2 \mathbf{1}^T G^2 \mathbf{1}\right].$$

Since $\|G+G^T\| = (2+o(1))np$ almost surely, therefore,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left(\frac{\rho}{(2+o(1))np}\right)^2 \mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1} - 4\mathbf{1}^T G^2\mathbf{1}\right]. \tag{D.20}$$

From Lemma 3,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right] = 2n(n-1)p(1-3p+2np).$$

From Lemma 4,

$$\mathbf{E}_G\left[\mathbf{1}^T G^2 \mathbf{1}\right] \geq n^2(n-2)p^2.$$

Therefore, from equation (D.20),

$$\mathbf{E}_G[R(\mathbf{p}_0)] \geq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \left(\frac{\rho}{(2+o(1))np}\right)^2 \left[2n(n-1)p(1-3p+2np) - 4n^2(n-2)p^2\right].$$

This implies that

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \Omega(p^{-1}) = \Omega\left(\frac{n}{c(n)\log n}\right).$$

Combining the upper bound of $\mathbf{E}_G[R(\mathbf{p}_0)]$, we conclude that

$$\mathbf{E}_G\left[R(\mathbf{p}_0)\right] = \Theta\left(\frac{n}{c(n)\log n}\right).$$

Since from equation (5.7), the optimal profit $\pi^* = \Theta(n)$. Therefore, the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \mathbf{E}_G\left[\frac{R(\mathbf{p}_0)}{\pi^*}\right] = \frac{\mathbf{E}_G[R(\mathbf{p}_0)]}{\Theta(n)} = \Theta\left(\frac{1}{c(n)\log n}\right).$$

Since we assume $p = \frac{c(n)\log n}{n}$, therefore our conclusion is equivalent to the statement in Theorem 5. □

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E. Proofs of Results for Random Networks with General Degree Distributions

E.1. Proof of Theorem 7

We first introduce the following lemma to provide a lower bound on the spectral norm of the matrix. We will use this lemma to prove the theorem.

LEMMA 5. *For any symmetric, non-negative $n \times n$ matrix A, assume that $i^* = \arg\max_{1 \le i \le n} \sum_{j=1}^n a_{ij}$, then $\|A\| \ge \sqrt{\sum_{j=1}^n a_{i^*j}^2}$.*

Proof By definition,

$$\|A\| = \max_{\|\xi\|=1} \xi^T A \xi.$$

The following inequality holds because adding more constraints leads to a lower bound of $\|A\|$.

$$\|A\| \ge \max_{\|\xi\|=1,\ \xi_j=0 \text{ for } j\in\{j: j\neq i^*, a_{i^*j}=0\}} \xi^T A \xi = \|A_1\|,$$

where A_1 is a $n \times n$ sub-matrix of A induced by i^* and its neighbors. By definition,

$$\|A_1\| = \max_{\|\eta\|=1} \eta^T A_1 \eta.$$

Since A_1 is a strongly connected graph, according to Perron-Frobenius Theorem, the eigenvector associated with the largest eigenvalue of A_1 is positive. This is equivalent to

$$\|A_1\| = \max_{\|\eta\|=1,\ \eta>0} \eta^T A_1 \eta.$$

Now we define A_2 as a sub-matrix of A_1, with all elements other than the i^*th row and i^*th column of A_1 being zero. Then we know that $A_1 \ge A_2$ and

$$\|A_1\| = \max_{\|\eta\|=1,\ \eta>0} \eta^T A_1 \eta \ge \max_{\|\eta\|=1,\ \eta>0} \eta^T A_2 \eta = \|A_2\|.$$

Since A_2 defines a star network, we know that

$$\|A_2\| = \sqrt{\sum_{j=1}^n a_{i^*j}^2}.$$

Combining all the above analysis, we know that $\|A\| \ge \sqrt{\sum_{j=1}^n a_{i^*j}^2}$. □

We now provide the complete proof of Theorem 7.

Proof Let $i^* = \arg\max_{1\le i\le n} \sum_{j=1}^n (G_{ij} + G_{ji})$. Since $(G + G^T)$ is a symmetric and non-negative matrix, by Lemma 5,

$$\|G + G^T\| \ge \sqrt{\sum_{j=1}^n (G_{i^*j} + G_{ji^*})^2}.$$

Since the adjacency matrix G is integer and non-negative, therefore,

$$\|G + G^T\| \ge \sqrt{\sum_{j=1}^n (G_{i^*j} + G_{ji^*})}.$$

From condition 1 in the theorem, almost surely,

$$\|G + G^T\| \ge \sqrt{\xi(n)}. \tag{E.1}$$

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Recall the upper bound of the expected regret from equation (D.5),

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\frac{\rho^2}{1-\rho}\left(\frac{1}{\|G+G^T\|}\right)^2 \mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right]. \tag{E.2}$$

From equation (E.1),

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\frac{1}{\xi(n)}\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right].$$

Now we compute the expected number of walks of length 2 in the network $(G+G^T)$. Each walk of length 2 includes one middle node and two of its possibly repeated edges. So the number walks of length 2 involving an arbitrary middle node v is $deg(v)^2$, where $deg(v)$ is the in/out-degree of node v in the network $(G+G^T)$. Therefore, the expected number of walks of length 2 in multigraph $(G+G^T)$ is

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G+G^T\right)^2\mathbf{1}\right] = n\mathbf{E}[d^2].$$

From condition 2 in the theorem, we have $\mathbf{E}(d^2) \leq \gamma(n)$. Therefore,

$$\mathbf{E}_G[R(\mathbf{p}_0)] \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{\rho^2}{1-\rho}\frac{n\gamma(n)}{\xi(n)} = O\left(\frac{n\gamma(n)}{\xi(n)}\right).$$

From equation (5.7), we know that $\pi^* = \Theta(n)$. Therefore, the expected fraction regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \mathbf{E}_G\left[\frac{R(\mathbf{p}_0)}{\pi^*}\right] = \frac{\mathbf{E}_G[R(\mathbf{p}_0)]}{\Theta(n)} = O\left(\frac{\gamma(n)}{\xi(n)}\right).$$

□

E.2. Proof of Theorem 8

Proof We apply Theorem 7 to prove the theorem. Particularly, the proof for different values of α is different. We use $a_1,...,a_n$ to denote the in-degree sequence and $b_1,...,b_n$ to denote the out-degree sequence.

Assume that X is a random variable following the distribution specified in equation (6.2). Then we can calculate the expectation of X as follows:

$$\mathbf{E}(X) = \left[x_{min}\frac{\alpha-1}{\alpha-2} - \frac{\alpha-1}{\alpha-2}\left(\frac{1}{x_{min}}\right)^{1-\alpha} n^{2-\alpha}\right] \Big/ \left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right).$$

Since $\alpha > 2$, we know that $\mathbf{E}(X) = \Theta(1)$.

Now we calculate the second moment of X. Different values of α lead to different results. Specifically, when $\alpha \neq 3$, the second moment is

$$\mathbf{E}(X^2) = \left[\frac{\alpha-1}{3-\alpha}\left(\frac{1}{x_{min}}\right)^{1-\alpha} n^{3-\alpha} - \frac{\alpha-1}{3-\alpha}x_{min}^2\right] \Big/ \left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right).$$

When $\alpha = 3$, the second moment is

$$\mathbf{E}(X^2) = \frac{\alpha-1}{(x_{min})^{1-\alpha}}\left[\log n - \log(x_{min})\right] \Big/ \left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right).$$

Therefore, depending on the value of α, the order of the second moment is different. In particular,

$$\mathbf{E}(X^2) = \begin{cases} \Theta(n^{3-\alpha}), & \text{if } 2 < \alpha < 3; \\ \Theta(\log n), & \text{if } \alpha = 3; \\ \Theta(1), & \text{if } \alpha > 3. \end{cases}$$

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In multigraph $(G+G^T)$, we calculate the expected degree as

$$\mathbf{E}(d_i)=\mathbf{E}(a_i)+\mathbf{E}(b_i)=2\mathbf{E}(a_i).$$

and the variance of the degree as

$$\mathbf{Var}(d_i)=\mathbf{Var}(a_i)+\mathbf{Var}(b_i)+2\mathbf{Cov}(a_i,b_i)=(2+2\rho_{a,b})\mathbf{Var}(a_i)=(2+2\rho_{a,b})(\mathbf{E}(a_i^2)-(\mathbf{E}(a_i))^2).$$

Therefore, the second moment of the degree distribution is given by:

$$\mathbf{E}(d_i^2)=\mathbf{Var}(d_i)+[\mathbf{E}(d_i)]^2=(2+2\rho_{a,b})\mathbf{E}(a_i^2)+(2-2\rho_{a,b})(\mathbf{E}(a_i))^2=\begin{cases}\Theta(n^{3-\alpha}), & \text{if } 2<\alpha<3;\\ \Theta(\log n), & \text{if } \alpha=3;\\ \Theta(1), & \text{if } \alpha>3.\end{cases}$$

Based on Theorem 7, we can utilize the bounds of the second moment of the degree distribution, and define

$$\gamma(n)=\begin{cases}\Theta(n^{3-\alpha}), & \text{if } 2<\alpha<3;\\ \Theta(\log n), & \text{if } \alpha=3;\\ \Theta(1), & \text{if } \alpha>3.\end{cases} \tag{E.3}$$

Now we calculate the bounds for the expected regret and expected fractional regret. By Law of Total Probability, we know that for any small enough $\delta>0$, the expected regret can be decomposed as

$$\begin{aligned}\mathbf{E}_G[R(\mathbf{p}_0)]=&\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i\ge x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]P\left(\max_{1\le i\le n}a_i\ge x_{min}n^{\frac{1-\delta}{\alpha-1}}\right)\\ &+\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i< x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]P\left(\max_{1\le i\le n}a_i< x_{min}n^{\frac{1-\delta}{\alpha-1}}\right).\end{aligned} \tag{E.4}$$

We first compute $\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i\ge x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]$. Because

$$\max_{1\le i\le n}\sum_{j=1}^n(G_{ij}+G_{ji})=\max_{1\le i\le n}(a_i+b_i)\ge\max_{1\le i\le n}a_i\ge x_{min}n^{\frac{1-\delta}{\alpha-1}},$$

we can choose $\xi(n)=x_{min}n^{\frac{1-\delta}{\alpha-1}}$.

Apply Theorem 7 and plug in equation (E.3), we obtain the following:

$$\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i\ge x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]=O\left(\frac{n\gamma(n)}{\xi(n)}\right)=\begin{cases}O\left(n^{4-\alpha-\frac{1-\delta}{\alpha-1}}\right), & \text{if } 2<\alpha<3;\\ O\left(n^{\frac{1+\delta}{2}}\log n\right), & \text{if } \alpha=3;\\ O\left(n^{\frac{\alpha-2+\delta}{\alpha-1}}\right), & \text{if } \alpha>3.\end{cases}$$

We then compute $\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i< x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]$. Because G is non-empty, we know that $\max_{1\le i\le n}\sum_{j=1}^n(G_{ij}+G_{ji})\ge 1$. Therefore we can define $\xi(n)=1$.

Applying Theorem 7 and pluging in equation (E.3), we obtain the following:

$$\mathbf{E}_G\left[R(\mathbf{p}_0)\middle|\max_{1\le i\le n}a_i< x_{min}n^{\frac{1-\delta}{\alpha-1}}\right]=O\left(\frac{n\gamma(n)}{\xi(n)}\right)=\begin{cases}O(n^{4-\alpha}), & \text{if } 2<\alpha<3;\\ O(n\log n), & \text{if } \alpha=3;\\ O(n), & \text{if } \alpha>3.\end{cases}$$

To complete the calculation of the expected regret, we calculate the probabilities in equation (E.4) as follows:

$$P\left(\max_{1\le i\le n}a_i<x_{min}n^{\frac{1-\delta}{\alpha-1}}\right)=\left[F\left(x_{min}n^{\frac{1-\delta}{\alpha-1}}\right)\right]^n=\left[1-\left(\frac{1}{n}\right)^{1-\delta}\right]^n\bigg/\left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right)^n=\Theta\left(e^{-n^\delta}\right).$$

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This also implies that

$$P\left(\max_{1\leq i\leq n} a_i \geq x_{min} n^{\frac{1-\delta}{\alpha-1}}\right) = 1 - \Theta\left(e^{-n^{\delta}}\right).$$

Therefore, based on equation (E.4), we can derive the bounds for expected regret for any small enough $\delta > 0$ as follows:

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \begin{cases} O(n^{4-\alpha-\frac{1-\delta}{\alpha-1}})\left(1-\Theta\left(e^{-n^{\delta}}\right)\right) + O(n^{4-\alpha})\Theta\left(e^{-n^{\delta}}\right) = O(n^{4-\alpha-\frac{1-\delta}{\alpha-1}}), & \text{if } 2 < \alpha < 3; \\ O(n^{\frac{1+\delta}{2}}\log n)\left(1-\Theta\left(e^{-n^{\delta}}\right)\right) + O(n\log n)\Theta\left(e^{-n^{\delta}}\right) = O(n^{\frac{1+\delta}{2}}\log n), & \text{if } \alpha = 3; \\ O\left(n^{\frac{\alpha-2+\delta}{\alpha-1}}\right)\left(1-\Theta\left(e^{-n^{\delta}}\right)\right) + O(n)\Theta\left(e^{-n^{\delta}}\right) = O\left(n^{\frac{\alpha-2+\delta}{\alpha-1}}\right), & \text{if } \alpha > 3. \end{cases}$$

The expected fractional regret is then given by

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \begin{cases} O(n^{3-\alpha-\frac{1-\delta}{\alpha-1}}), & \text{if } 2 < \alpha < 3; \\ O(n^{-\frac{1-\delta}{2}}\log n), & \text{if } \alpha = 3; \\ O\left(n^{-\frac{1-\delta}{\alpha-1}}\right), & \text{if } \alpha > 3. \end{cases}$$

The above expressions for expected regret and expected fractional regret is equivalent to the following: the expected regret

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \begin{cases} O(n^{4-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2 < \alpha < 3; \\ O(n^{\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3; \\ O\left(n^{\frac{\alpha-2}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

and the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \begin{cases} O(n^{3-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2 < \alpha < 3; \\ O(n^{-\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3; \\ O\left(n^{-\frac{1}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

for any $\epsilon > 0$.

Since ϵ can take any small enough positive value, this further implies that the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = o(1)$. □

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