

# REVEALED PRICE PREFERENCE: THEORY AND EMPIRICAL ANALYSIS

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SEPTEMBER 20, 2018

ABSTRACT. With the aim of determining the welfare implications of price change in consumption data, we introduce a revealed preference relation over prices. We show that an absence of cycles in this preference relation characterizes a model of demand where consumers trade-off the utility of consumption against the disutility of expenditure. This model is appropriate whenever a consumer’s demand over a *strict* subset of all available goods is being analyzed. For the random utility extension of the model, we devise non-parametric statistical procedures for testing and welfare comparisons. The latter requires the development of novel tests of linear hypotheses for partially identified parameters. In doing so, we provide new algorithms for the calculation and statistical inference in non-parametric counterfactual analysis for a general partially identified model. Our applications on national household expenditure data provide support for the model and yield informative bounds concerning welfare rankings across different prices.

## 1. INTRODUCTION

Suppose we observe a consumer’s purchases of two foodstuffs, fresh produce and pizza, from two separate trips to a grocery store. In the first instance  $t$ , the prices are  $p^t = (1, 1)$  of produce and pizza respectively and she buys a bundle  $x^t = (3, 1)$ . In her second trip  $t'$ , the prices are  $p^{t'} = (2, 1)$  and she purchases  $x^{t'} = (1, 4)$ . Is the consumer better off at observation  $t$  or  $t'$ ?

The standard way of trying to answer this question is to check for revealed preference. Was  $x^t$  cheaper than  $x^{t'}$  when  $x^{t'}$  was purchased? If so,  $x^{t'}$  is revealed preferred to  $x^t$ .

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We are very grateful for the comments provided by the seminar participants at the Cowles Foundation Conferences (April 2015, June 2015), University of Toronto (March 2016), Heterogeneity in Supply and Demand Conference at Boston College (December 2016), Identification of Rationality in Markets and Games Conference at Warwick University (March 2017), University of Cambridge (March 2017), UCSD (April 2017), UCLA (April 2017), the Canadian Economic Theory Conference (May 2017), “Meeting in Econometrics” at TSE (May 2017), Iowa State University (September 2017) and Boston University (October 2017). We also thank Samuel Norris and Krishna Pendakur for generously sharing their data with us. Deb thanks the SSHRC for their continued financial support and the Cowles foundation for their hospitality. Stoye acknowledges support from NSF grant SES-1260980 and through Cornell University’s Economics Computer Cluster Organization, which was partially funded through NSF Grant SES-0922005. We thank Matthew Thirkettle for excellent research assistance.

Similarly, we can ask whether  $x^t$  is revealed preferred to  $x^{t'}$ . In this case neither bundle is revealed preferred to the other since  $p^t x^t = 4 < 5 = p^t x^{t'}$  and  $p^{t'} x^{t'} = 6 < 7 = p^{t'} x^t$ . Indeed, it is not difficult to see that there is a utility function (defined on produce and pizza) rationalizing the data in which  $x^t$  gives higher utility than  $x^{t'}$ , and another rationalization in which  $x^{t'}$  gives higher utility than  $x^t$ .

But there is another way of interpreting this data. Casual observation suggests that the consumer is better off at  $t$  as an increase in the price of produce causes her to substitute towards the less nutritious food item (pizza). But even ignoring the labels of the goods, it is clear the consumer is better off at  $t$  compared to  $t'$  as the prices at  $t'$  are uniformly higher (keeping fixed other exogenous aspects of the economic environment); were  $p^t$  and  $p^{t'}$  the prices at two grocery stores then there is no question as to which one she would choose. Thus it seems natural to make welfare comparisons by endowing the consumer with a *preference over prices*. Furthermore, this preference can be manifested even when price vectors are not ordered across all goods: the consumer should prefer  $p^t$  over  $p^{t'}$  whenever the bundle  $x^{t'}$  chosen at  $p^{t'}$  could have been purchased at the prices  $p^t$  at lower cost. This is because the consumer always has the option of choosing  $x^{t'}$  at observation  $t$ , and this choice would leave her with money that could potentially be used for other purposes. Formally, if at observations  $t$  and  $t'$ , we find that  $p^t x^{t'} \leq (<) p^{t'} x^{t'}$ ,

*the consumer has revealed that she (strictly) prefers the price  $p^t$  to the price  $p^{t'}$ .*

This concept of revealed preference recognizes that there are alternative uses to money besides the goods observed in the data and any change in prices that leads to lower expenditure on the same bundle must make the consumer better off.

The first aim of this paper is to build a new theory of consumer behavior based on this simple observation. We then characterize how this theory manifests itself in cross-sectional consumer demand data. Based on this result, we develop and implement a novel econometric method for determining the population-wide welfare implications of price changes using the model.

There are four reasons why we think the above goal is worth pursuing.

(1) *The problem we pose is a natural analog of the standard problem studied in Afriat (1967).*

The standard revealed preference theory of the consumer is built around the generalized axiom of revealed preference (GARP), which states that the revealed preference over demand bundles must obey a no-cycling condition. This condition, which is imposed on a finite data set  $\mathcal{D} = \{(p^t, x^t)\}_{t \in T}$  of price-demand observations, is a necessary implication of the transitivity of a consumer's preference; the essential content of [Afriat's Theorem](#) is that it is also sufficient for  $\mathcal{D}$  to be rationalized by a strictly increasing utility function.

But data sets typically employed in empirical analysis involve only observations of a *subset* of the consumer's purchases. As we have argued above, such data can also generate

another natural revealed preference relation—that over prices. Analogously to GARP, in order for welfare comparisons to be consistent, we would have to require that the revealed preference over prices is free of cycles, a property we call the *generalized axiom of price preference* (GAPP). This leads inevitably to the following question: precisely what does GAPP mean for consumer behavior?

(2) *Our axiom has an intuitive utility representation.*

We show that GAPP is both necessary and sufficient for the existence of a strictly increasing function  $U : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  such that

$$x^t \in \operatorname{argmax}_{x \in \mathbb{R}_+^L} U(x, -p^t x) \text{ for all } t = 1, 2, \dots, T.$$

The function  $U$  should be interpreted as an *expenditure-augmented utility function*, where  $U(x, -e)$  is the consumer's utility when she acquires  $x$  at the cost of  $e$ .<sup>1</sup> It recognizes that the consumer's expenditure on the observed goods is endogenous and dependent on prices: she could in principle spend more than what she actually spent (note that she optimizes over  $x \in \mathbb{R}_+^L$ ) but the trade-off is the disutility of greater expenditure.

There are applications for which the behavior captured by our model has normative appeal. For instance, if we were studying a dataset containing supermarket purchases (typically, a strict subset of total consumption), it would be odd to assume that a consumer goes to the store with an exogenously determined budget; instead, it is more natural to expect that the amount she ends up spending depends on the prices. Of course, spending money on groceries has an opportunity cost: more money spent in the supermarket means less money available for other unobserved goods. The expenditure-augmented utility above captures this in a simple reduced form way via its last argument. We feel that this simplicity is a strength of the model as it seems reasonable to assume that the typical shopper uses a simple heuristic to determine the opportunity cost of a marginal purchase, even when this behavior is not exactly globally optimizing if she takes into account the prices of all other goods (besides groceries) that is relevant to her welfare.

That said, it is possible to give the expenditure-augmented utility function a standard interpretation by situating it within a larger optimization problem for the consumer. (We provide the most basic example here, others can be found in [Section 2.3](#).) Indeed, a consumer who maximizes  $U$  is indistinguishable from one who is maximizing a (conventional) utility function  $G : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}$  defined over the  $L$  observed goods plus an outside good, with the consumer's permitted expenditure on all  $L + 1$  goods held fixed at some number  $M > 0$  (unobserved by the econometrician) and with the price of the

<sup>1</sup>In the main part of the paper, we allow for the consumption space to be a discrete subset of  $\mathbb{R}_+^L$  and for prices to be nonlinear.

outside good held constant across observations;<sup>2</sup> in other words, normalizing the price of the outside good at 1, the consumer maximizes  $G(x, z)$  subject to  $px + z \leq M$ . Notice that if  $G(x, z) = \bar{U}(x) + z$  and the consumer chooses  $(x, z)$  to maximize  $G$  subject to  $px + z \leq M$ , then the optimal  $x$  will also maximize  $U(x, -p^t x) := \bar{U}(x) - px$ . This is the familiar quasilinear case, which is commonly used in partial equilibrium analysis, both to model demand and to carry out welfare analysis of price changes. Thus, we could think of the expenditure-augmented model as a generalization of the quasilinear model.

Of course, even when only a subset of goods is observed, one could directly test GARP (and almost inevitably this is what is done in empirical applications). For this test to be justified, one must assume that the consumer has a subutility defined on the  $L$  observed goods that is weakly separable from all other goods. In other words, the consumer is maximizing a ‘broader’ utility function  $G(x, y) = H(\bar{U}(x), y)$ , where  $y$  is the vector of all other goods relevant to the consumer’s wellbeing and  $H$  is some increasing aggregator function. Then at any time  $t$ , as the consumer chooses some bundle that maximizes her overall utility, the sub-bundle  $x^t$  must necessarily maximize  $\bar{U}(x)$ , conditional on the expenditure devoted to those goods, i.e., for all  $x$  such that  $p^t x \leq p^t x^t$ . This in turn guarantees that  $\mathcal{D} = \{(p^t, x^t)\}_{t \in T}$  obeys GARP.<sup>3</sup>

Notice that it is certainly possible for a data set  $\mathcal{D}$  to obey both GARP and GAPP; indeed, one such example is data generated by a consumer who maximizes a quasilinear utility function  $U(x, -p^t x) = \bar{U}(x) - p^t x$ .<sup>4</sup> Both axioms represent distinct restrictions on a data set (see Section 2.5) based on different, but *not* mutually exclusive, theories of optimizing behavior and it is generally unhelpful to think of them as competing alternatives. Whether we appeal to the restrictions based on one theory or the other depends in part on the nature of the data (for example, whether the goods are such that the weak separability assumption is valid in the case of GARP), but most of all it depends on the objective of our econometric analysis.

### 3. *Our model allows for consistent welfare comparisons at different prices.*

There is a sense in which the GARP-based model is very robust. As we have pointed out, it makes sense only if weak separability is imposed, but once that is in place, the validity of its notion of rationalization is completely independent of what happens outside the  $L$  observed goods. It is possible, over the course of the data collection period, for the

<sup>2</sup>Ignoring boundary issues, a consumer who chooses  $x$  to maximize  $U(x, -px)$  is equivalent to one who chooses  $(x, z)$  to maximize  $G(x, z)$ , subject to  $px + z \leq M$ , where  $G(x, z) := U(x, -M + z)$ .

<sup>3</sup>Strictly speaking under the additional assumption that either  $\bar{U}$  is locally nonsatiated or that there is a perfectly divisible outside good and  $H$  is strictly increasing in that good (Polisson and Quah, 2013).

<sup>4</sup>More generally, GARP and GAPP will both hold if the consumer chooses  $x^t$  to maximize an expenditure-augmented utility function of the form  $H(\bar{U}(x), -e)$ , i.e., where there is a sub-utility function  $\bar{U}$  defined on the  $L$  goods.

prices of unobserved goods to change or for the consumer to inherit an unexpected large fortune, and she should still be maximizing the subutility conditional on the observed expenditure level  $p^t x^t$  (even though the expenditure level itself will typically vary with external factors). A manifestation of this is that the GARP-based model only hinges on the relative prices among the  $L$  goods: if GARP holds in a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t \in T}$  then it holds in the data set  $\mathcal{D}' = \{(\lambda^t p^t, x^t)\}_{t \in T}$ , where  $\lambda^t$  are positive scalars. The inevitable flip side of this characteristic of the model is that it cannot tell whether one price situation  $p^t$  is better or worse than another  $p^{t'}$ .<sup>5</sup>

On the other hand, prices clearly have cardinal content in the GAPP-based model and indeed the whole model is constructed with the aim of comparing prices consistently. The inevitable downside of this feature is that it is only reasonable under *ceteris paribus*-type assumptions. If the consumer should receive a large surprise fortune during the data collection period, it may no longer be plausible to assume that her expenditure augmented utility function is unchanging, and the same may be true of large fluctuations in the prices of the non-observed goods. Thus we may wish to restrict applications of our model to data over a period where it is reasonable to believe that the augmented utility function is stable, perhaps after adjusting observed prices by a suitable deflator (to account for changing prices on the outside goods). In any case, GAPP can be tested on the data and, if it passes, the model allows us to compare the consumer's wellbeing under different prices on the observed goods.

#### 4. *Welfare bounds can be inferred from repeated cross-sectional data.*

After formulating the expenditure-augmented utility model and exploring its theoretical features, the second part of our paper is devoted to demonstrating how the model can be used for empirical welfare analysis. This first involves generalizing our model to a population of consumers (as our empirical application employs repeated cross-sectional data) and then developing a test of the model. Assuming that the data passes the test, we could then estimate the proportion of consumers who are made better or worse off by a given change in prices. It is worth pointing out that the econometric theory we develop for this latter step is itself a stand alone contribution and has more general applications beyond the empirical exercise in this paper.

Although we test the random (expenditure-augmented) utility extension of our model, we could in principle test the basic model on panel data sets of household or individual demand (for example, from purchasing information collected in scanner panels).<sup>6</sup> The

<sup>5</sup>This is no longer true if the GARP model is imposed, not on  $\mathcal{D}$  but on an extended data set including a representative good which covers unobserved goods, but this brings along its own complications (see part (3) in Section 2.3).

<sup>6</sup>Tests and applications of the standard model using [Afriat's Theorem](#) or its extensions are very common (for a recent survey, see [Crawford and De Rock \(2014\)](#)) and our model is just as straightforward to test on

random utility extension is useful for at least three reasons: (i) It is more general in that it allows for individual preferences in the population to change over time, provided the population distribution of preferences stays the same. (ii) It makes no restriction on unobserved preference heterogeneity and does not require any demand aggregation. (iii) It allows us to demonstrate that our framework can accommodate sampling uncertainty inherent in cross-sectional data (applications to panel data are more direct) and so can be applied on a wider range of datasets.

The random utility extension of the GARP-based model is due to [McFadden and Richter \(1991\)](#). They assume that the econometrician observes the distribution of demand choices on each of a finite number of budget sets and characterize the observable content of this model, under the assumption that the distribution of preferences is stable across observations. There are two main challenges that must be overcome before their model can be taken to data. First, [McFadden and Richter \(1991\)](#) do not account for finite sample issues as they assume that the econometrician observes the population distributions of demand; this hurdle was recently overcome by [Kitamura and Stoye \(2017\)](#) who develop a testing procedure for the GARP-based model which incorporates sampling error. Second, the test suggested by [McFadden and Richter \(1991\)](#) requires the observation of large samples of consumers who face not only the same prices but also make *identical expenditures*, a feature which is not true of any observational data. Thus to implement the test they suggest, [Kitamura and Stoye \(2017\)](#) need to first estimate demand distributions at a fixed level of (median) expenditure, which requires the use of an instrumental variable technique (with all its attendant assumptions) to adjust for the endogeneity of observed expenditure.

A remarkable feature of the random utility version of our (GAPP-based) model is that it can be tested directly on data, *even when the demand distribution at a given price vector are associated with different expenditure levels*. This allows us to estimate the demand distribution by simply using sample frequencies and we can avoid the additional layer of demand estimation needed to test the GARP-based model. The reason for this is somewhat ironical: we show that the data set is consistent with our model if and only if it could be converted (in a specific sense) into a data set of the type (with identical expenditures at each price) envisaged by [McFadden and Richter \(1991\)](#) and passes the test suggested by them. In other words, we apply the test suggested by [McFadden and Richter \(1991\)](#), but not for the model they have in mind. This also means that we can use, and in a more straightforward way, the econometric techniques in [Kitamura and Stoye \(2017\)](#).

Assuming that a data set is consistent with our model, we can then evaluate the welfare impact of an observed change in prices. Indeed, if we observe the true distribution of

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a panel data set. It is also common to test the standard model in experimental settings. Note that, our test is *not* appropriate on typical experimental data where the budget is provided exogenously as part of the experiment (and is not endogenous as in our model).

demand at each price, it is possible to impose bounds on the proportion of the population who are revealed better off or worse off by an observed change in prices. Of course, when samples are finite, these bounds instead have to be estimated. To do so, we develop new econometric techniques that allow us to form confidence intervals on the proportion of consumers who are better or worse off; these techniques build on the econometric theory in Kitamura and Stoye (2017) but are distinct from it. We note that these techniques can be more generally applied to linear hypothesis testing of parameter vectors that are partially identified, even in models that are unrelated to demand theory (see, for example, Lazziati, Quah, and Shirai (2018)). More importantly, we provide a new general method for estimation and inference in nonparametric counterfactual analysis. As the evaluation of counterfactuals is an important goal of empirical research, this paper offers a potentially useful practical tool for practitioners.

Finally, we test, and find support for, the expenditure augmented utility model on two separate national household expenditure data sets from Canada and the U.K. We also estimate the welfare impact of the changes in observed prices. We demonstrate that these could be meaningfully estimated, even though they are typically only partially identified: the estimated bounds on the percentage of households who are made better or worse off by a price change are almost always narrower than 10% and often substantially narrower than that.

### 1.1. Organization of the paper

The remainder of this paper is structured as follows. Section 2 lays out the deterministic model and its revealed preference characterization, and Section 3 generalizes it to our analog of a random utility model. Section 4 describes the nonparametric statistical test of our model and the novel econometric theory needed to estimate the welfare bounds. Section 5 illustrates the econometric techniques with an empirical application, and Section 6 concludes. The appendix contains proofs that are not in the main body of the paper.

## 2. THE DETERMINISTIC MODEL

The primitive in the analysis in this section is a data set of a single consumer's purchasing behavior collected by an econometrician. The econometrician observes the consumer's purchasing behavior over  $L$  goods and the prices at which those goods were chosen. In formal terms, the bundle is in  $\mathbb{R}_+^L$  and the prices are in  $\mathbb{R}_{++}^L$  and so an observation at  $t$  can be represented as  $(p^t, x^t) \in \mathbb{R}_{++}^L \times \mathbb{R}_+^L$ . The *data set* collected by the econometrician is  $\mathcal{D} := \{(p^t, x^t)\}_{t=1}^T$ . We will slightly abuse notation and use  $T$  both to refer to the number of observations, which we assume is finite, and the set  $\{1, \dots, T\}$ ; the meaning will be clear from the context. Similarly,  $L$  could denote both the number, and the set, of commodities.

Before motivating and presenting our model, it is worth providing a short description of the standard theory of revealed preference. A locally nonsatiated<sup>7</sup> utility function  $\tilde{U} : \mathbb{R}_+^L \rightarrow \mathbb{R}$  is said to *rationalize* a data set  $\mathcal{D}$  if

$$x^t \in \operatorname{argmax}_{\{x \in \mathbb{R}_+^L : p^t x \leq p^t x^t\}} \tilde{U}(x) \quad \text{for all } t \in T. \quad (1)$$

Standard *revealed preference* is captured by two binary relations,  $\succeq_x$  and  $\succ_x$  which are defined on the chosen bundles observed in  $\mathcal{D}$ , that is, the set  $\mathcal{X} := \{x^t\}_{t \in T}$  as follows:

$$x^{t'} \succeq_x (\succ_x) x^t \text{ if } p^{t'} x^{t'} \geq (>) p^{t'} x^t.$$

We say that the bundle  $x^{t'}$  is *directly revealed (strictly) preferred* to  $x^t$  if  $x^{t'} \succeq_x (\succ_x) x^t$ , that is, whenever the bundle  $x^t$  is (strictly) cheaper at prices  $p^{t'}$  than the bundle  $x^{t'}$ . This terminology is, of course, very intuitive. If the agent is maximizing some locally nonsatiated utility function  $\tilde{U} : \mathbb{R}_+^L \rightarrow \mathbb{R}$ , then if  $x^{t'} \succeq_x x^t$  ( $x^{t'} \succ_x x^t$ ), it must imply that  $\tilde{U}(x^{t'}) \geq (>) \tilde{U}(x^t)$ .

We denote the transitive closure of  $\succeq_x$  by  $\succeq_x^*$ , that is, for  $x^{t'}$  and  $x^t$  in  $\mathcal{X}$ , we have  $x^{t'} \succeq_x^* x^t$  if there are  $t_1, t_2, \dots, t_N$  in  $T$  such that  $x^{t'} \succeq_x x^{t_1}$ ,  $x^{t_1} \succeq_x x^{t_2}$ ,  $\dots$ ,  $x^{t_{N-1}} \succeq_x x^{t_N} \succeq_x x^t$ , and  $x^{t_N} \succeq_x x^t$ ; in this case, we say that  $x^{t'}$  is *revealed preferred* to  $x^t$ . If anywhere along this sequence, it is possible to replace  $\succeq_x$  with  $\succ_x$  then we say that  $x^{t'}$  is *revealed strictly preferred* to  $x^t$  and denote that relation by  $x^{t'} \succ_x^* x^t$ . Once again, this terminology is completely natural since if  $\mathcal{D}$  is rationalizable by some locally nonsatiated utility function  $\tilde{U}$ , then  $x^{t'} \succeq_x^* (\succ_x^*) x^t$  implies that  $\tilde{U}(x^{t'}) \geq (>) \tilde{U}(x^t)$ . This observation in turn implies that a necessary condition for rationalization is that the revealed preference relation has no cycles.

**Definition 2.1.** A data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  satisfies the *Generalized Axiom of Revealed Preference* or *GARP* if there do not exist two observations  $t, t' \in T$  such that  $x^{t'} \succeq_x^* x^t$  and  $x^t \succ_x^* x^{t'}$ .

The main insight of [Afriat's Theorem](#) is to show that this condition is also sufficient (the formal statement can be found in the online [Appendix C](#)).

### 2.1. Consistent welfare comparisons across prices

Typically, the  $L$  goods whose demand is being monitored by the econometrician constitute no more than a part of the purchasing decisions made by the consumer. The consumer's true budget (especially when one takes into account the possibility of borrowing and saving) is never observed and the expenditure which she devotes to the  $L$  goods is a

<sup>7</sup>This means that at any bundle  $x$  and open neighborhood of  $x$ , there is a bundle  $y$  in the neighborhood with strictly higher utility.



decision made by the consumer and is dependent on the prices. The consumer's choice over the  $L$  goods inevitably affects what she could spend on, and therefore her consumption of, other goods not observed by the econometrician. Given this, the rationalization criterion (1) used in [Afriat's Theorem](#) will only make sense under an additional assumption of weak separability: the consumer has a sub-utility function  $\bar{U}$  defined over the  $L$  goods and the utility function of the consumer, defined over all goods, takes the form  $H(\bar{U}(x), y)$ , where  $x$  is the bundle of  $L$  goods observed by the econometrician and  $y$  is the bundle of unobserved goods. Assuming that the consumer chooses  $(x, y)$  to maximize  $G(x, y) = H(\bar{U}(x), y)$ , subject to a budget constraint  $px + qy \leq M$  (where  $M$  is her wealth and  $p$  and  $q$  are the prices of the observed and unobserved goods respectively), then, at the prices  $p^t$ , the consumer's choice  $x^t$  will obey (1) provided  $H$  is strictly increasing in either the first or second argument. This provides the theoretical motivation to test for the existence of a sub-utility function  $\bar{U}$  that rationalizes a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ . Notice that once the weak separability assumption is in place, the background requirements of Afriat's test are very modest in the sense that it is possible for prices of the unobserved goods and the unobserved total wealth to change arbitrarily across observations, without affecting the validity of the test. This is a major advantage in applications but the downside is that the conclusions of this model are correspondingly limited to ranking different bundles among the observed goods via the sub-utility function.

Now suppose that instead of checking for rationalizability in the sense of (1), the econometrician would like to ask a different question: given a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ , can he sign the welfare impact of a price change from  $p^t$  to  $p^{t'}$ ? Stated differently, this question asks whether he can compare the consumer's *overall* utility after a change in the prices of the  $L$  goods from  $p^t$  and  $p^{t'}$ , holding fixed other aspects of the economic environment that may affect the consumer's welfare, such as the prices of unobserved goods and her overall wealth. Perhaps the most basic welfare comparison in this setting can be made as follows: if at prices  $p^{t'}$ , the econometrician finds that  $p^{t'}x^t < p^tx^t$  then he can conclude that the agent is better off at the price vector  $p^{t'}$  compared to  $p^t$ . This is because, at the price  $p^{t'}$  the consumer can, if she wishes, buy the bundle bought at  $p^t$  and she would still have money left over to buy other things, so she must be strictly better off at  $p^{t'}$ . This ranking is eminently sensible, but can it lead to inconsistencies?

**Example 1.** Consider a two observation data set

$$p^t = (2, 1), x^t = (4, 0) \text{ and } p^{t'} = (1, 2), x^{t'} = (0, 1).$$

which is depicted in [Figure 1](#). Given that the budget sets do not even cross, we know that GARP holds. Since  $p^{t'}x^t < p^tx^t$ , it seems that we may conclude that the consumer is better off at prices  $p^{t'}$  than at  $p^t$ ; however, it is also true that  $p^{t'}x^{t'} > p^tx^{t'}$ , which gives the opposite conclusion.

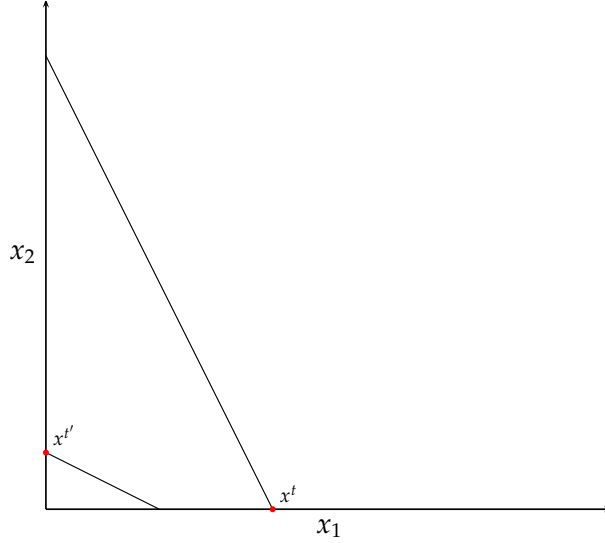


FIGURE 1. Choices that do not allow for consistent welfare predictions.

This example shows that for an econometrician to be able to consistently compare the consumer's welfare at different prices, some restriction (different from GARP) has to be imposed on the data set. To be precise, define the binary relations  $\succeq_p$  and  $\succ_p$  on  $\mathcal{P} := \{p^t\}_{t \in T}$ , that is, the set of price vectors observed in  $\mathcal{D}$ , in the following manner:

$$p^{t'} \succeq_p (\succ_p) p^t \text{ if } p^{t'} x^t \leq (<) p^t x^t.$$

We say that price  $p^{t'}$  is *directly (strictly) revealed preferred* to  $p^t$  if  $p^{t'} \succeq_p (\succ_p) p^t$ , that is, whenever the bundle  $x^t$  is (strictly) cheaper at prices  $p^{t'}$  than at prices  $p^t$ . We denote the transitive closure of  $\succeq_p$  by  $\succeq_p^*$ , that is, for  $p^{t'}$  and  $p^t$  in  $\mathcal{P}$ , we have  $p^{t'} \succeq_p^* p^t$  if there are  $t_1, t_2, \dots, t_N$  in  $T$  such that  $p^{t'} \succeq_p p^{t_1}$ ,  $p^{t_1} \succeq_p p^{t_2}, \dots, p^{t_{N-1}} \succeq_p p^{t_N}$ , and  $p^{t_N} \succeq_p p^t$ ; in this case we say that  $p^{t'}$  is *revealed preferred* to  $p^t$ . If anywhere along this sequence, it is possible to replace  $\succeq_p$  with  $\succ_p$  then we say that  $p^{t'}$  is *revealed strictly preferred* to  $p^t$  and denote that relation by  $p^{t'} \succ_p^* p^t$ . Then the following restriction is the bare minimum required to exclude the possibility of circularity in the econometrician's assessment of the consumer's wellbeing at different prices.

**Definition 2.2.** The data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  satisfies the *Generalized Axiom of Price Preference* or *GAPP* if there do not exist two observations  $t, t' \in T$  such that  $p^{t'} \succeq_p^* p^t$  and  $p^t \succ_p^* p^{t'}$ .

This in turn leads naturally to the following question: if a consumer's observed demand behavior obeys GAPP, what could we say about her decision making procedure?

## 2.2. The Expenditure-Augmented Utility Model

An *expenditure-augmented utility function* (or simply, an *augmented utility function*) is a utility function that has, as its arguments, both the bundle consumed by the consumer  $x$  and the total expense  $e$  incurred in acquiring the bundle. Formally, the augmented utility function  $U$  has domain  $\mathbb{R}_+^L \times \mathbb{R}_-$ , where  $U(x, -e)$  is assumed to be strictly increasing in the last argument; in other words, utility is strictly decreasing in expenditure. This second argument captures the opportunity cost of money and is a simple, reduced form way of modeling the tradeoff with all other financial decisions made by the consumer. At a given price  $p$ , the consumer chooses a bundle  $x$  to maximize  $U(x, -px)$ . We denote the *indirect utility at price  $p$*  by

$$V(p) := \sup_{x \in \mathbb{R}_+^L} U(x, -px). \quad (2)$$

If the consumer's augmented utility maximization problem has a solution at every price vector  $p \in \mathbb{R}_{++}^L$ , then  $V$  is also defined at those prices and this induces a reflexive, transitive, and complete preference over prices in  $\mathbb{R}_{++}^L$ .

A data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  is *rationalized by an augmented utility function* if there exists such a function  $U : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  with

$$x^t \in \operatorname{argmax}_{x \in \mathbb{R}_+^L} U(x, -p^t x) \quad \text{for all } t \in T.$$

Notice that unlike the notion of rationalization in [Afriat's Theorem](#), we do not require the bundle  $x$  to be chosen from the budget set  $\{x \in \mathbb{R}_+^L : p^t x \leq p^t x^t\}$ . The consumer can instead choose from the entire consumption space  $\mathbb{R}_+^L$ , though the utility function penalizes expenditure.

It is straightforward to see that GAPP is necessary for a data set to be rationalized by an augmented utility function. First, notice that if  $p^{t'} \succeq_p p^t$ , then  $p^{t'} x^t \leq p^t x^t$ , and so

$$V(p^{t'}) \geq U(x^t, -p^{t'} x^t) \geq U(x^t, -p^t x^t) = V(p^t).$$

Furthermore,  $U(x^t, -p^{t'} x^t) > U(x^t, -p^t x^t)$  if  $p^{t'} \succ_p p^t$ , and in that case  $V(p^{t'}) > V(p^t)$ . Suppose GAPP were not satisfied and there were two observations  $t, t' \in T$  such that  $p^{t'} \succeq_p^* p^t$  and  $p^t \succ_p^* p^{t'}$ . Then there would exist  $t_1, t_2, \dots, t_N \in T$  such that

$$V(p^{t'}) \geq V(p^{t_1}) \geq \dots \geq V(p^{t_N}) \geq V(p^t) > V(p^{t'})$$

which is impossible.

Our main theoretical result, which we state next, also establishes the sufficiency of GAPP for rationalization. Moreover, the result states that whenever  $\mathcal{D}$  can be rationalized, it can be rationalized by an augmented utility function  $U$  with a list of properties that make it convenient for analysis. In particular, we can guarantee that there is always

a solution to  $\max_{x \in \mathbb{R}_+^L} U(x, -p \cdot x)$  for any  $p \in \mathbb{R}_{++}^L$ . This property guarantees that  $U$  will generate preferences over all the price vectors in  $\mathbb{R}_{++}^L$ . Clearly, it is also necessary for making out-of-sample predictions and, indeed, it is important for the internal consistency of the model.<sup>8</sup>

**Theorem 1.** *Given a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ , the following are equivalent:*

- (1)  $\mathcal{D}$  can be rationalized by an augmented utility function.
- (2)  $\mathcal{D}$  satisfies GAPP.
- (3)  $\mathcal{D}$  can be rationalized by an augmented utility function  $U$  that is strictly increasing, continuous, and concave. Moreover,  $U$  is such that  $\max_{x \in \mathbb{R}_+^L} U(x, -p \cdot x)$  has a solution for all  $p \in \mathbb{R}_{++}^L$ .

REMARK: In order to keep the presentation simple, our maintained assumption throughout this paper is that the consumption space is  $\mathbb{R}_+^L$ . However, this theorem (and the rest of the theory which builds on it) applies also to the case where the consumption space is some closed subset  $X$  of  $\mathbb{R}_+^L$ ; indeed, our proof covers this more general case. For example, if we model all the observed goods as discrete, then  $X = \mathbb{N}_+^L$ . Another possibility is that the consumer is deciding on buying a subset of objects from a set  $A$  with  $L$  items. Then each subset could be represented as an element of  $X = \{0, 1\}^L$ ; for  $x \in X$ , the  $i$  entry ( $x_i$ ) equals 1 if and only if the  $i$ th object is chosen.

When the consumption space is  $X$ , the notion of a rationalization will obviously have to be altered to require the optimality, in  $X$  rather than in  $\mathbb{R}_+^L$ , of the chosen bundle. Retaining our definitions of the revealed preference relations  $\succeq_p$  and  $\succ_p$ , it is straightforward to check that GAPP is still necessary for the maximization of an augmented utility function. Indeed, the whole theorem remains essentially valid, with statement (3) modified to read: “ $\mathcal{D}$  can be rationalized by an augmented utility function  $U : X \times \mathbb{R}_- \rightarrow \mathbb{R}$  with the following properties: (i) it admits an extension  $\widehat{U} : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  that is strictly increasing, continuous, and concave; (ii)  $\max_{x \in X} U(x, -px)$  has a solution for all  $p \in \mathbb{R}_{++}^L$ .”

**PROOF OF THEOREM 1.** We will show that (2)  $\implies$  (3). We have already argued that (1)  $\implies$  (2) and (3)  $\implies$  (1) by definition.

Choose a number  $M > \max_t p^t x^t$  and define the augmented data set  $\widetilde{\mathcal{D}} = \{(p^t, 1), (x^t, M - p^t x^t)\}_{t=1}^T$ . This data set augments  $\mathcal{D}$  since we have introduced an  $L + 1$ th good, which we have priced at 1 across all observations, with the demand for this good equal to  $M - p^t x^t$ .

The crucial observation to make here is that

$$(p^t, 1)(x^t, M - p^t x^t) \geq (p^t, 1)(x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^{t'} x^{t'} \geq p^t x^t,$$

<sup>8</sup>Suppose the data set is rationalized but only by an augmented utility function for which the existence of an optimum is not generally guaranteed, then it undermines the hypothesis tested since it is not clear why the sample collected should then have the property that an optimum exists.

which means that

$$(x^t, M - p^t x^t) \succeq_x (x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^t \succeq_p p^{t'}.$$

Similarly,

$$(p^t, 1)(x^t, M - p^t x^t) > (p^{t'}, 1)(x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^{t'} x^{t'} > p^t x^t,$$

and so

$$(x^t, M - p^t x^t) \succ_x (x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^t \succ_p p^{t'}.$$

Consequently,  $\mathcal{D}$  satisfies GAPP if and only if  $\tilde{\mathcal{D}}$  satisfies GARP. Applying [Afriat's Theorem](#) (see Remark 3 in [Appendix C](#)), when  $\tilde{\mathcal{D}}$  satisfies GARP, there is  $\tilde{U} : \mathbb{R}^{L+1} \rightarrow \mathbb{R}$  such that

$$(x^t, M - p^t x^t) \in \underset{\{x \in \mathbb{R}_+^L : p^t x + m \leq M\}}{\operatorname{argmax}} \tilde{U}(x, m) \quad \text{for all } t \in T.$$

The function  $\tilde{U}$  can be chosen to be strictly increasing, continuous, and concave, and the lower envelope of a finite set of affine functions. Note that augmented utility function  $\bar{U} : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  defined by  $\bar{U}(x, -e) := \tilde{U}(x, M - e)$  rationalizes  $\mathcal{D}$  as  $x^t$  solves  $\max_{x \in \mathbb{R}_+^L} \bar{U}(x, -p^t x)$  by construction. Furthermore,  $\bar{U}$  is strictly increasing in  $(x, -e)$ , continuous, and concave.

Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a differentiable function with  $h(0) = 0$ ,  $h'(k) > 0$ ,  $h''(k) \geq 0$  for  $k \in \mathbb{R}_+$ , and  $\lim_{k \rightarrow \infty} h'(k) = \infty$ .<sup>9</sup> Define  $\hat{U} : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  by

$$\hat{U}(x, -e) := \bar{U}(x, -e) - h(\max\{0, e - M\}). \quad (3)$$

It is clear that this function is strictly increasing in  $(x, -e)$ , continuous, and concave. Furthermore,  $x^t$  solves  $\max_{x \in \mathbb{R}_+^L} \hat{U}(x, -p^t x)$  and hence solves  $\max_{x \in X} \hat{U}(x, -p^t x)$ . This is because  $\hat{U}(x, -e) \leq \bar{U}(x, -e)$  for all  $(x, -e)$ , and  $\hat{U}(x^t, -p^t x^t) = \bar{U}(x^t, -p^t x^t)$ .

Lastly, we claim that at every  $p \in \mathbb{R}_{++}^L$ ,  $\operatorname{argmax}_{x \in X} \hat{U}(x, -px)$  is nonempty. Choose a sequence  $x^n \in X$  such that  $\hat{U}(x^n, -px^n)$  tends to  $\sup_{x \in X} \hat{U}(x, -px)$  (which we allow to be infinity). It is impossible for  $px^n \rightarrow \infty$  because the piecewise linearity of  $U(x, -e)$  in  $x$  and the assumption that  $\lim_{k \rightarrow \infty} h'(k) \rightarrow \infty$  implies that  $\hat{U}(x^n, -px^n) \rightarrow -\infty$ . So the sequence  $px^n$  is bounded, which in turn means that there is a subsequence of  $x^n$  that converges to  $x^* \in X$  (since  $X$  is closed). By the continuity of  $\hat{U}$ , we obtain  $\hat{U}(x^*, -px^*) = \sup_{x \in X} \hat{U}(x, -px)$ . Lastly, note that the restriction of  $\hat{U}$  to  $X \times \mathbb{R}_-$  still rationalizes  $\mathcal{D}$ . ■

From this point onwards, when we refer to ‘rationalization’ without additional qualifiers, we shall mean rationalization by an augmented utility function, that is, in the sense established by [Theorem 1](#) rather than in the sense established by [Afriat's Theorem](#).

<sup>9</sup>For example,  $h(k) = k^3$ .

### 2.3. Interpretations of expenditure-augmented utility and related models

The inclusion of expenditure in the augmented utility function captures the opportunity cost incurred by the consumer when she chooses to buy some bundle of goods. In this section we will discuss various issues and interpretations of this model.

#### (1) Constrained optimization on a larger problem

The proof of [Theorem 1](#) itself provides an interpretation of the opportunity cost of expenditure. Suppose that the consumer is maximizing an overall utility function that depends both on the observed bundle  $x$  and on a bundle  $y$  of other goods, subject to a global budget of  $M$ ; in formal terms, the consumer is maximizing the overall utility  $G(x, y)$  subject to  $px + qy \leq M$ , where  $q$  are the prices of other goods. Keeping  $q$  and  $M$  fixed, we can then interpret  $U(x, -e)$  as the greatest overall utility the consumer can achieve by choosing  $y$  optimally, subject to expenditure  $M - e$  and conditional on consuming  $x$ , i.e.,

$$U(x, -e) = \max_{\{y \geq 0 : qy \leq M - e\}} G(x, y). \quad (4)$$

In this case, expenditure on the observed goods incurs an opportunity cost since it diminishes what is available for the unobserved goods.

Obviously, a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  collected from this consumer will obey GAPP since  $(x^t, y^t)$  maximizes  $G(x, y)$  subject to  $px + qy \leq M$  if and only if  $x^t$  maximizes  $U(x, -p^t x)$ . If the econometrician could also observe  $y^t$ , then he will find that the data set  $\{((p^t, q), (x^t, y^t))\}_{t=1}^T$  obeys GARP, but  $\mathcal{D}$  will typically not obey GARP.

In the case where the data is collected over an extended period, the assumption that outside prices are fixed at  $q$  may be problematic, but under certain assumptions it is possible to preserve GAPP when prices of the observed goods are suitably deflated.

#### (2) Constrained optimization on a larger problem, with changes in the external environment

Suppose that the prices of the outside goods are allowed to vary during the data collection period. We consider two types of price variation. The first is the case where the external prices may move up or down proportionately.<sup>10</sup> Thus at observation  $t$ , the external prices are  $\lambda^t q$ , for some scalar  $\lambda^t > 0$ . Suppose also that the agent's global budget at time  $t$  increases by a factor of  $\lambda^t$ . The interpretation of this is that expenditure on the observed goods represent just a small fraction of the consumer's wealth, the larger portion of which is devoted to the outside goods; thus a change of wealth by the same factor  $\lambda^t$  means that the agent's nominal wealth is keeping pace with price inflation (of the outside goods). At observation  $t$ , the consumer maximizes  $G(x, y)$  subject to  $(x, y)$  obeying

$$p^t x + \lambda^t q y \leq \lambda^t M.$$

<sup>10</sup>Note that this is the case considered in the Hicks Aggregation Theorem.

Dividing this inequality by  $\lambda^t$ , we see that the consumer's choice must be identical to the case where the price of the observed goods is  $p^t/\lambda^t$ , with the external prices and total wealth constant at  $q$  and  $M$  respectively. It follows that the data collected from this consumer will obey GAPP, provided the prices are appropriately deflated, i.e., the set  $\{(p^t/\lambda^t, x^t)\}_{t=1}^T$  will obey GAPP.

In the case where prices of the outside goods are completely free to vary, it is still possible for GAPP to hold after deflating prices, but stronger assumptions would need to be imposed on  $G$ . The following is an example. Suppose that the outside goods are weakly separable from the observed goods, so the utility function has the form  $G(x, y) = H(x, u(y))$ , where  $u$  is the sub-utility on the outside goods. Furthermore, we assume that  $u$  has an indirect utility  $v$  which is of the Gorman polar form, i.e., at the price vector  $q \gg 0$  (of the outside goods) and expenditure  $w$ ,

$$v(q, w) = a(q)w + b(q),$$

where  $a$  is homogenous of degree -1 and  $b$  is homogeneous of degree 0. The global budget is assumed to vary in such a way that, should the consumer devote all of her wealth to the unobserved goods, then the indirect utility is constant at  $M$ ; in other words, the agent's real wealth (as measured by the indirect utility function  $v$ ) is unchanged across the observations. This implies that  $M^t = [M - b(q^t)]/a(q^t)$ . The consumer chooses  $(x, v)$  to maximize  $G(x, v)$  subject to

$$p^t x + \frac{v - b(q^t)}{a(q^t)} \leq \frac{M - b(q^t)}{a(q^t)}.$$

Multiplying this inequality by  $a(q^t)$ , we see that it is equivalent to maximizing  $G(x, v)$  subject to  $a(q^t)p^t x + v \leq M$ . Thus the data set  $\{(a(q^t)p^t, x^t)\}_{t=1}^T$  will obey GAPP.

It is worth emphasizing that while the conditions we give in (1) and (2) may provide motivation for the augmented utility model in terms of a larger setup that some readers will find more familiar, it is *not* necessary for the validity of our approach. Even when those conditions do not hold exactly, we think it is reasonable to directly assume that the consumer has a preference over bundles of the observed  $L$  goods and their associated expenditure, which the consumer has developed as a habit and which guides her purchasing decisions (over the period where observations are taken). Bear in mind that no one is required to have an unconditional belief in this model: it suffices that it is a plausible hypothesis that we are willing to bring to the data, and then whether our belief is confirmed (or disproved) will depend on the outcome of the GAPP test.

(3) *Assessing the impact of price changes by enlarging the optimization problem*

Our discussion in (1) and (2) suggests an alternative approach to comparing the impact of price changes. Instead of appealing to GAPP, one could simply add a good, which represents all outside goods, to the  $L$  observed goods. Pricing this outside good at  $q^t$  and then assuming that the total expenditure on all  $L + 1$  goods  $M^t$  is observable, one could then test GARP on the data set  $\{(p^t, q^t), (x^t, M^t - p^t x)\}_{t=1}^T$ . Assuming that this set passes GARP, it would be rationalizable by some utility function defined on  $L + 1$  goods; the welfare impact of different prices of the observed goods (fixing the total expenditure and the price of the representative outside good) could then be compared via the indirect utility function. Something akin to this approach is common in the empirical literature.<sup>11</sup>

Comparing our approach with this alternative, notice that in both cases, there is a need to form an aggregate/composite good across the outside goods and to give it a price index; in other words, if outside prices are changing in significant ways, then a full justification of this type of aggregation will require, in *both* approaches, assumptions of the type discussed in (2). The crucial difference between the two approaches lies in how the global budget term  $M$  is handled. In the alternative approach, it is assumed that  $M$  is known and can be measured across the data set; in practice, it is often identified with annual income. In our approach, if we choose to interpret the augmented utility function as arising from a larger constrained optimization problem (as in (1) and (2)), then we are assuming that the global budget is constant in real terms, but we do not explicitly identify (or link) it with income or any other observable.

There are several reasons why we think our approach is at least a useful alternative. Obviously, compared with this approach, ours is useful when information on annual income is not available (which is a feature of many commonly used data sets). But even when this information is available, it is plainly not the right value to take as the global budget if the consumer can save and borrow to a significant degree (as acknowledged, for example, in Hausman and Newey (2016)). More generally, figuring out what really constitutes the global budget on which the consumer is basing her purchasing decisions on the  $L$  goods and their alternatives is not straightforward, especially if the consumer engages in ‘mental budgeting’ (see Thaler (1999)). Indeed, in addition to the large evidence from laboratory data, there is recent empirical evidence from field data that shows that money is not fungible and that consumers often keep separate accounts (typically not observable to the econometrician) for different categories of purchases (see, for instance, Feldman (2010), Hastings and Shapiro (2013) and Milkman and Beshears (2009)).

<sup>11</sup>For recent examples, see Blundell, Horowitz, and Parey (2012) and Hausman and Newey (2016). In both these papers, the empirical application involves the case where  $L = 1$  (specifically, the good examined is gasoline, so it is a two-good demand system when the numeraire is included); when  $L = 1$ , our approach imposes no meaningful restrictions on data so it does not readily provide a viable alternative way to study the specific empirical issues in those papers.



(4) *Quasilinear utility and other special cases*

The augmented utility function could be thought of as a generalization of the *quasilinear* model. In this case, the consumer derives a utility of  $\bar{U}(x)$  from the bundle  $x$  and maximizes utility net of expenditure, that is, she chooses a bundle  $x$  that maximizes

$$U(x, -e) := \bar{U}(x) - e. \quad (5)$$

This formulation can be motivated by assuming that the consumer has an overall utility function that is quasilinear, that is,  $G(x, y) = \bar{U}(x) + y$ , where there is one unobserved representative good consumed at level  $y \in \mathbb{R}_+$ . If we normalize the price of the outside good at 1 and assume that  $M$  is sufficiently large, then the consumer maximizes  $G(x, y)$  subject to  $px + y \leq M$  if and only if he chooses  $x$  to maximize  $U(x, -px) = \bar{U}(x) - px$ . Notice that there is a complete absence of wealth effects in this model; fluctuations in the global budget  $M$  has no impact on the demand for the observed goods and there is neither a need to observe  $M$  nor to assume that it is constant in real terms in a data set. This partly accounts for its widespread use in partial equilibrium analysis, both to model demand and to carry out welfare analysis of price changes.

The revealed preference characterization for the quasilinear model is known (see [Brown and Calsamiglia \(2007\)](#)) so it is possible to use this model as a way of evaluating non-parametrically the welfare impact of price changes (after a suitable deflation of the outside prices). However, this model is considerably more restrictive than the augmented utility model and it has behavioral implications which go beyond the absence of revealed price preference cycles. For instance, suppose  $L = 2$  and the consumer at prices  $(p_1, p_2) \in \mathbb{R}_{++}^2$  prefers the bundle  $(a, b)$  to another bundle  $(b, a)$ ; then it is straightforward to check that this preference is maintained at the prices  $(p_1 + k, p_2 + k) \in \mathbb{R}_{++}^2$  for any  $k$ .

The quasilinear model is a model where the agent has a sub-utility over the  $L$  observed goods.<sup>12</sup> More generally, this holds for any model expressible in the following way:

$$U(x, -e) = H(\bar{U}(x), -e), \quad (6)$$

where  $H$  is some increasing aggregator function. Any data set collected from an agent maximizing an augmented utility function of this form will obey both GAPP and GARP. The latter holds because if  $x^t$  maximizes  $U(x, -p^t x)$  then it must also be maximizing  $\bar{U}(x)$  subject to  $x \in \{x \in \mathbb{R}_+^L : p^t x \leq p^t x^t\}$ .

When the augmented utility function has the form in (6), the agent's preference over bundles when expenditure is held fixed is independent of the level of expenditure, i.e., if  $U(x'', -e) \geq (>) U(x', -e)$  for some  $e > 0$ , then  $U(x'', -\bar{e}) \geq (>) U(x', -\bar{e})$  for any  $\bar{e} > 0$ . On the other hand, the general augmented utility model does not require a sub-utility

<sup>12</sup>Note that this is distinct from the example in (2) where the agent has a sub-utility on the *unobserved* goods.

over the  $L$  goods and thus the preference between bundles is allowed to change with its value.<sup>13</sup> This greater flexibility is significant because it accommodates phenomena which we may not wish to exclude a priori. For example, suppose the first good is food and the second alcohol, and the consumer is indifferent between  $(10, 5)$  and  $(5, 10)$  when both bundles are valued at 100, i.e.,  $U((100, 50), -100) = U((50, 100), -100)$ . If the agent has a sub-utility over food and alcohol, then this indifference is preserved if both bundles are valued at 150. However, the higher expenditure incurred in acquiring the bundle will mean less money available for other activities such as entertainment or socializing. If alcohol and entertainment are substitutes, then this could mean that alcohol becomes more desirable for the consumer, so that  $U((100, 50), -150) < U((50, 100), -150)$ ; if this occurs,  $U$  cannot have the form (6).

#### 2.4. Local robustness of the GAPP test

The augmented utility function captures the idea that a consumer's choice is guided by the satisfaction she derives from the bundle as well as the opportunity cost of acquiring that bundle, as measured by its expense. Price data may contain measurement error which implies that prices of the observed goods  $p^t$  we use for testing may well differ from their true values  $q^t$ . Put differently, at time  $t$ , the consumer actually maximizes  $U(x, -q^t x)$  instead of  $U(x, -p^t x)$  which is the hypothesis we are testing.

Another potential source of error is that the agent's unobserved wealth or mental budget may change from one observation to the next and this could manifest itself as a change in the consumer's stock of the representative outside good. If this happens, then at observation  $t$ , the consumer would be maximizing  $U(x, -p^t x + \delta^t)$ , where  $\delta^t$  is the perturbation in wealth at time  $t$ , instead of maximizing  $U(x, -p^t x)$ . Of course, the errors could potentially enter simultaneously in both prices and wealth.

Statement (2) of [Proposition 1](#) below shows that inference from a GAPP test is unaffected as long as the size of the errors is bounded (equation (7) provides the specific bound). Specifically, so long as  $\mathcal{D}$  obeys a mild genericity condition (which is satisfied in our empirical application), the GAPP test is locally robust in the sense that any conclusion obtained through the test remains valid for sufficiently small perturbations of the original hypothesis. For example, a data set that fails GAPP is not consistent with the maximization of an augmented utility function, for any prices sufficiently close to the ones observed and after allowing for small wealth perturbations.

<sup>13</sup>If the augmented utility function is differentiable, then the marginal rate of substitution between goods 1 and 2 is given by  $(\partial U / \partial x_1) / (\partial U / \partial x_2)$  evaluated at  $(x, -e)$ . This will in general depend on  $e$ , though not in the case where  $U$  has the form (6).

Not surprisingly, if the errors are allowed to be unbounded, then the model ceases to have any content (statement (1) of [Proposition 1](#)). Specifically, we can always find wealth perturbations  $\delta^t$  (while prices  $p^t$  are assumed to be measured without error) such that each  $x^t$  maximizes  $U(x, -p^t x + \delta^t)$ , with no restrictions on  $\mathcal{D}$ . In particular, these perturbations can be chosen to be mean zero.

**Proposition 1.** *Given a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ , the following hold:*

- (1) *There exists an augmented utility function  $U$  and  $\{\delta^t\}_{t=1}^T$ , with  $\sum_{t=1}^T \delta^t = 0$ , such that  $x^t \in \operatorname{argmax}_{x \in \mathbb{R}_+^L} U(p^t, -p^t x^t + \delta^t)$  for all  $t \in T$ .*
- (2) *Suppose,  $\mathcal{D}$  satisfies  $p^t x^t - p^{t'} x^t \neq 0$  for all  $t \neq t'$  and let  $\{q^t\}_{t=1}^T$  and  $\{\delta^t\}_{t=1}^T$  satisfy*

$$2 \max_{t \in T} \{|\delta^t|\} + 2B \max_{t \in T, i \in L} \{|\epsilon_i^t|\} < \min_{t, t' \in T, t \neq t'} |p^t x^t - p^{t'} x^t| \quad (7)$$

where  $B = \max_{t \in T} \{\sum_{i=1}^L |x_i^t|\}$  and  $\epsilon_i^t := q_i^t - p_i^t$ .

*If  $\mathcal{D}$  obeys GAPP, there is an augmented utility function  $U$  such that*

$$x^t \in \operatorname{argmax}_{x \in \mathbb{R}_+^L} U(q^t, -q^t x^t + \delta^t) \text{ for all } t \in T, \quad (8)$$

*and if  $\mathcal{D}$  violates GAPP, then there is no augmented utility function  $U$  such that (8) holds.*

Additionally, [Aguiar and Kashaev \(2017\)](#) have recently developed a technique to explicitly account for measurement error in revealed preference tests. Their method can, in principle, also be applied to our model (as the test of our model can be equivalently written in the first order condition form they require).

## 2.5. Comparing GAPP and GARP

Recall that [Example 1](#) in [Section 2.1](#) is an example of a data set that obeys GARP but fails GAPP. We now present an example of a data set that obeys GAPP but fails GARP.

**Example 2.** Consider the data set consisting of the following two choices:

$$p^t = (2, 1), x^t = (2, 1) \text{ and } p^{t'} = (1, 4), x^{t'} = (0, 2).$$

These choices are shown in [Figure 2](#). This is a classic GARP violation as  $p^t x^t = 5 > 2 = p^{t'} x^t$  ( $x^t \succ_x x^{t'}$ ) and  $p^{t'} x^{t'} = 8 > 6 = p^t x^{t'}$  ( $x^{t'} \succ_x x^t$ ). In words, each bundle is strictly cheaper than the other at the budget set corresponding to the latter observation. However, these choices satisfy GAPP as  $p^{t'} x^{t'} = 8 > 2 = p^t x^{t'}$  ( $p^t \succ_p p^{t'}$ ) but  $p^t x^t = 5 \not\geq 6 = p^{t'} x^t$  ( $p^{t'} \not\prec_p p^t$ ).

The upshot of this example is that there are data sets which admit rationalization with an augmented utility function but cannot be rationalized in Afriat's sense (as in (1)). If we interpret the augmented utility function in the form  $G(x, y)$  (given by (4)), then this

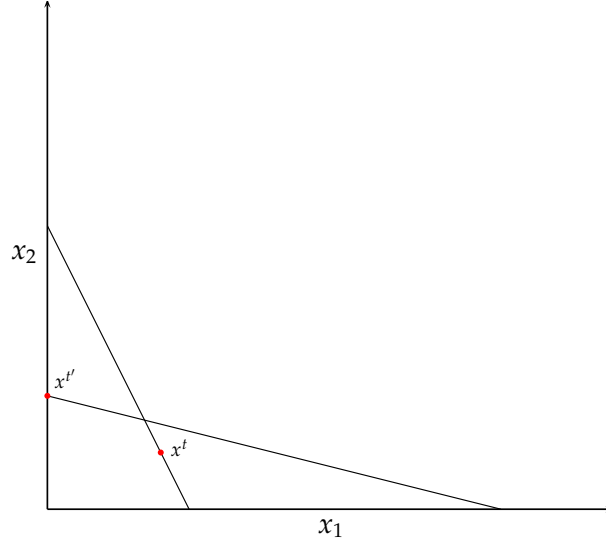


FIGURE 2. Choices that satisfy GAPP but not GARP

means that while the agent's behavior is consistent with the maximization of an overall utility function  $G$ , this utility function is not weakly separable in the observed goods  $x$ . In particular, this implies that the agent does not have a quasilinear utility function (with  $G(x, y) = U(x) + y$ ), which is weakly separable in the observed goods and will generate data sets obeying both GAPP and GARP.

While GAPP and GARP are not in general the same conditions, they coincide in any data set where  $p^t x^t = 1$  for all  $t \in T$ . This is because  $x^t \succeq_x x^{t'}$  if and only if  $p^t \succeq_p x^{t'}$  (since both conditions are equivalent to  $1 \geq p^t x^{t'}$ ) and, similarly,  $x^t \succ_x x^{t'}$  if and only if  $p^t \succ_p x^{t'}$ . Given a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ , we define the *expenditure-normalized* version of  $\mathcal{D}$  as another data set  $\check{\mathcal{D}} := \{(p^t, \check{x}^t)\}_{t=1}^T$ , such that  $\check{x}^t = x^t / (p^t x^t)$ . This new data set has the feature that  $p^t \check{x}^t = 1$  for all  $t \in T$ . Notice that the revealed price preference relations  $\succeq_p, \succ_p$  remain unchanged when consumption bundles are scaled. Thus a data set obeys GAPP if and only if its normalized version also obeys GAPP, which in this case is equivalent to GARP. The next proposition gives a more detailed statement of these observations.

**Proposition 2.** *Let  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  be a data set and let  $\check{\mathcal{D}} = \{(p^t, \check{x}^t)\}_{t \in T}$ , where  $\check{x}^t = x^t / (p^t x^t)$ , be its expenditure-normalized version. Then the revealed preference relations  $\succeq_p^*$  and  $\succ_p^*$  on  $\mathcal{P} = \{p^t\}_{t=1}^T$  and the revealed preference relations  $\succeq_x^*$  and  $\succ_x^*$  on  $\check{\mathcal{X}} = \{\check{x}^t\}_{t=1}^T$  are related in the following manner:*

- (1)  $p^t \succeq_p^* p^{t'}$  if and only if  $\check{x}^t \succeq_x^* \check{x}^{t'}$ .
- (2)  $p^t \succ_p^* p^{t'}$  if and only if  $\check{x}^t \succ_x^* \check{x}^{t'}$ .

*As a consequence,  $\mathcal{D}$  obeys GAPP if and only if  $\check{\mathcal{D}}$  obeys GARP.*

*Proof.* Notice that

$$p^t \frac{x^t}{p^t x^t} \geq p^{t'} \frac{x^{t'}}{p^{t'} x^{t'}} \iff p^{t'} x^{t'} \geq p^t x^t.$$

The left side of the equivalence says that  $\check{x}^t \succeq_x \check{x}^{t'}$  while the right side says that  $p^t \succeq_p p^{t'}$ . This implies (1) since  $\succeq_p^*$  and  $\succeq_x^*$  are the transitive closures of  $\succeq_p$  and  $\succeq_x$  respectively. Similarly, it follows from

$$p^t \frac{x^t}{p^t x^t} > p^{t'} \frac{x^{t'}}{p^{t'} x^{t'}} \iff p^{t'} x^{t'} > p^t x^t$$

that  $\check{x}^t \succ_x \check{x}^{t'}$  if and only if  $p^t \succ_p p^{t'}$ , which leads to (2). The claims (1) and (2) together guarantee that there is a sequence of observations in  $\mathcal{D}$  that lead to a GAPP violation if and only if the analogous sequence in  $\check{\mathcal{D}}$  lead to a GARP violation.  $\square$

As an illustration, compare the data sets in [Figure 1](#) and [Figure 2](#) to the expenditure-normalized data sets in [Figure 3a](#) and [Figure 3b](#). It can be clearly observed that the expenditure-normalized data in [Figure 3a](#) contains a GARP violation (which implies it does not satisfy GAPP) whereas the data in [Figure 3b](#) does not violate GARP (and, hence, satisfies GAPP).

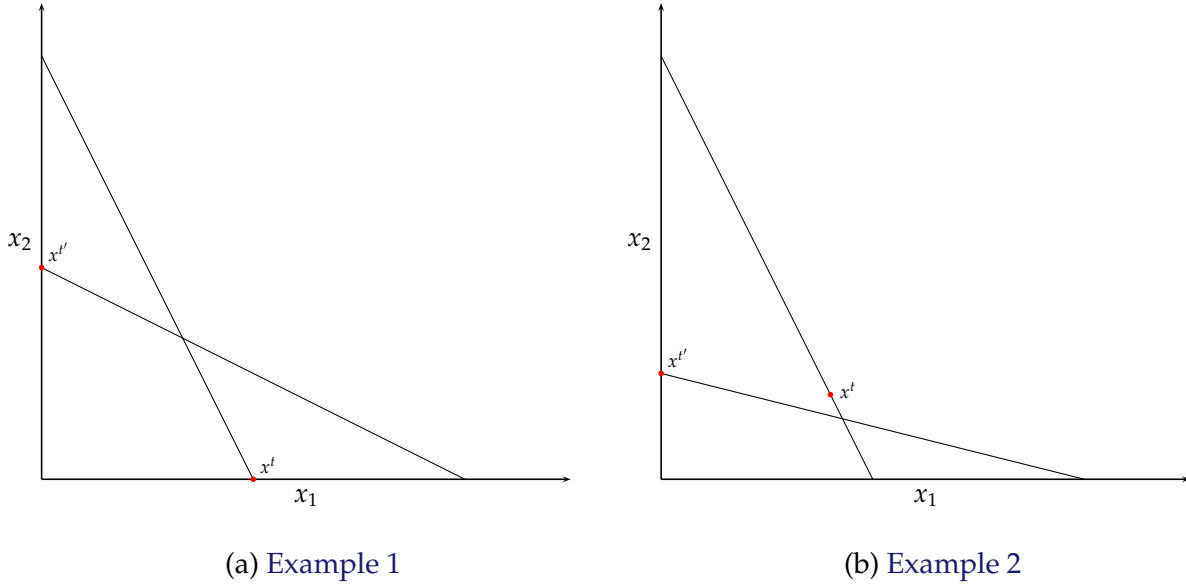


FIGURE 3. Expenditure-Normalized Choices

A consequence of [Proposition 2](#) is that the expenditure-augmented utility model can be tested in two ways: we can either test GAPP directly or we can normalize the data by expenditure and then test GARP (equivalently GAPP). If we are simply interested in testing GAPP on a data set  $\mathcal{D}$ , normalization brings no advantage: the test is computationally straightforward in either case and involves the construction of their (respective) revealed

preference relations and checking for acyclicity. However, as we shall see in the final section, the possibility of testing GAPP after expenditure normalization plays a crucial role in guaranteeing that there is a convenient test for the augmented utility model when we have cross-sectional data on demand.<sup>14</sup>

## 2.6. Preference over Prices

We know from [Theorem 1](#) that if  $\mathcal{D}$  obeys GAPP then it can be rationalized by an augmented utility function with an indirect utility that is defined at all price vectors in  $\mathbb{R}_{++}^L$ . It is straightforward to check that any indirect utility function  $V$  as defined by (2) has the following two properties:

- (a) it is *nonincreasing in  $p$* , in the sense that if  $p' \geq p$  (in the product order) then  $V(p') \leq V(p)$ , and
- (b) it is *quasiconvex in  $p$* , in the sense that if  $V(p) = V(p')$ , then  $V(\beta p + (1 - \beta)p') \leq V(p)$  for any  $\beta \in [0, 1]$ .

Any rationalizable data set  $\mathcal{D}$  could potentially be rationalized by many augmented utility functions and each one of them will lead to a different indirect utility function. We denote this set of indirect utility functions by  $\mathbf{V}(\mathcal{D})$ . We have already observed that if  $p^t \succeq_p^* (\succ_p^*) p^{t'}$  then  $V(p^t) \geq (>) V(p^{t'})$  for any  $V \in \mathbf{V}(\mathcal{D})$ ; in other words, the conclusion that the consumer prefers the prices  $p^t$  to  $p^{t'}$  is *fully nonparametric* in the sense that it is independent of the precise augmented utility function used to rationalize  $\mathcal{D}$ . The next result says that, without further information on the agent's augmented utility function, this is *all* the information on the agent's preference over prices in  $\mathcal{P}$  that we could glean from the data. Thus, in our nonparametric setting, the revealed price preference relation contains the most detailed information for welfare comparisons.

**Proposition 3.** *Suppose  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  is rationalizable by an augmented utility function. Then for any  $p^t, p^{t'}$  in  $\mathcal{P}$ :*

- (1)  $p^t \succeq_p^* p^{t'}$  if and only if  $V(p^t) \geq V(p^{t'})$  for all  $V \in \mathbf{V}(\mathcal{D})$ .
- (2)  $p^t \succ_p^* p^{t'}$  if and only if  $V(p^t) > V(p^{t'})$  for all  $V \in \mathbf{V}(\mathcal{D})$ .

<sup>14</sup>We note that there is an analogous ‘GARP-version’ of [Proposition 2](#) and that this observation (or some close variation of it) has been exploited before in the literature (see, for example, ([Sakai, 1977](#))). Suppose  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  obeys GARP. Then GARP holds even if each observed price vector  $p^t$  is arbitrarily scaled. In particular,  $\mathcal{D}$  obeys GARP if and only if  $\hat{\mathcal{D}} = \{(\hat{p}^t, x^t)\}_{t \in T}$ , where  $\hat{p}^t = p^t / (p^t x^t)$ , obeys GARP, and equivalently GAPP since  $\hat{p}^t x^t = 1$  for all  $t \in T$ . The latter perspective is useful because it highlights the possibility of applying Afriat's Theorem on  $\hat{\mathcal{D}}$ , *in the space of prices* (in other words, with the roles of prices and bundles reversed). This immediately gives us a different, ‘dual’ rationalization of  $\mathcal{D}$  in terms of indirect utility, i.e., there is a continuous, strictly decreasing, and convex function  $v : \mathbb{R}_{++}^L \rightarrow \mathbb{R}$  such that  $\hat{p}^t \in \arg \min_{\{p \in \mathbb{R}_{++}^L : p x^t \geq 1\}} v(p)$ . For an application of this observation see, for example, [Brown and Shannon \(2000\)](#).

### 2.7. Extension to nonlinear pricing

We have so far assumed that prices are linear, but this feature is not crucial to our main result. Given that goods are sometimes priced nonlinearly (for instance, quantity discounts, bundle pricing etc.), and this feature can be important in certain modeling contexts, it would be useful to have a version of [Theorem 1](#) that allows for that possibility.

In the linear case, the good  $i$  has price  $p_i$  and the cost of a bundle  $x$  is  $px$ . More generally, we define a price system as a map  $\psi : \mathbb{R}_+^L \rightarrow \mathbb{R}_+$ , where  $\psi(x)$  is interpreted as the cost to the consumer of the bundle  $x$ . We require  $\psi$  to be continuous, strictly increasing (in the sense that  $\psi(x'') > \psi(x')$  whenever  $x'' > x'$ ), and that, for any number  $M > 0$ ,  $\psi^{-1}(M)$  is a bounded set. The last condition means that the consumer could not acquire an arbitrarily large bundle with finite expenditure. We assume that both the *price system* and the bundle chosen by the consumer are observed, so that a data set is formally a collection  $\mathcal{D} = \{(\psi^t, x^t)\}_{t=1}^T$ . This data set is rationalized by an expenditure augmented utility function  $U : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  if  $x^t \in \arg \max_{x \in \mathbb{R}_+^L} U(x, -\psi^t(x))$  for all  $t \in T$ .

The notion of revealed preference over prices can be extended to a revealed preference over price systems. Let  $\mathcal{P} := \{\psi^t\}_{t \in T}$ . We say that price  $\psi^{t'}$  is *directly revealed preferred* (*directly revealed strictly preferred*) to  $\psi^t$  if  $\psi^{t'}(x^t) \leq (<) \psi^t(x^t)$ ; we denote this by  $\psi^{t'} \succeq_p$  ( $\succ_p$ )  $\psi^t$ .

As in the linear case, we denote the transitive closure of  $\succeq_p$  by  $\succeq_p^*$ , that is,  $\psi^{t'} \succeq_p^* \psi^t$  if there are  $t_1, t_2, \dots, t_N$  in  $T$  such that  $\psi^{t'} \succeq_p \psi^{t_1}$ ,  $\psi^{t_1} \succeq_p \psi^{t_2}$ ,  $\dots$ ,  $\psi^{t_{N-1}} \succeq_p \psi^{t_N}$ , and  $\psi^{t_N} \succeq_p \psi^t$ ; in this case we say that  $\psi^{t'}$  is *revealed preferred* to  $\psi^t$ . If anywhere along this sequence, it is possible to replace  $\succeq_p$  with  $\succ_p$  then we denote that relation by  $\psi^{t'} \succ_p^* \psi^t$  and say that  $\psi^{t'}$  is *strictly revealed preferred* to  $\psi^t$ . It is straightforward to check that if  $\mathcal{D}$  can be rationalized by an expenditure augmented utility function then it obeys the following generalization of GAPP to nonlinear price systems:

*there do not exist observations  $t, t' \in T$  such that  $\psi^{t'} \succeq_p^* \psi^t$  and  $\psi^t \succ_p^* \psi^{t'}$ .*

The following result says that the converse is also true. The proof uses the extension of [Afriat's Theorem](#) to nonlinear prices and general consumption spaces<sup>15</sup> found in [Nishimura, Ok, and Quah \(2017\)](#).

**Theorem 2.** *Given a data set  $\mathcal{D} = \{(\psi^t, x^t)\}_{t=1}^T$ , the following are equivalent:*

- (1)  $\mathcal{D}$  can be rationalized by an augmented utility function.
- (2)  $\mathcal{D}$  satisfies nonlinear GAPP.
- (3)  $\mathcal{D}$  can be rationalized by an augmented utility function  $U$  that is strictly increasing and continuous, such that  $\max_{x \in \mathbb{R}_+^L} U(x, -\psi(x))$  has a solution for any price system  $\psi$ .

<sup>15</sup>As with [Theorem 1](#), the proof is for the more general case where the consumption space is a subset  $X \subset \mathbb{R}_+^L$ .

### 3. THE STOCHASTIC MODEL

In this section, we develop the stochastic version of the expenditure-augmented utility model. We begin by explaining the corresponding extension of the Afriat model, as found in [McFadden and Richter \(1991\)](#), [McFadden \(2005\)](#) (henceforth to be referred to as [MR](#)).

#### 3.1. Rationalization by Random Utility

Suppose that instead of observing single choices on  $T$  budget sets, the econometrician observes choice probabilities on each budget set. Our preferred interpretation is that each observation corresponds to the distribution of choices made by a population of consumers and the data set consists of a repeated cross section of such choice probabilities. This interpretation is appropriate for our empirical application which uses repeated cross-sectional data from a population.

We denote the budget set corresponding to observation  $t$  by  $B^t := \{x \in \mathbb{R}_+^L : p^t x = 1\}$ . In this model, only relative prices matter, so we can scale prices and normalize income to 1 without loss of generality. We use  $\hat{\pi}^t$  to denote the probability measure of choices on budget set  $B^t$  at observation  $t$ . Thus, for any subset  $X^t \subset B^t$ ,  $\hat{\pi}^t(X^t)$  denotes the probability that the choices lie in the subset  $X^t$ . Following [MR](#) and [Kitamura and Stoye \(2017\)](#) (henceforth, referred to as [KS](#)), we assume that the econometrician observes the stochastic data set  $\tilde{\mathcal{D}} := \{(B^t, \hat{\pi}^t)\}_{t=1}^T$ , which consists of a finite collection of budget sets along with the corresponding choice probabilities. Note that, in practice,  $\hat{\pi}^t$  needs to be estimated.

For ease of exposition, we also impose the following assumption on the data.

**Assumption 1.** For all  $t, t' \in T$  with  $B^t \neq B^{t'}$ , the choice probabilities satisfy  $\hat{\pi}^t(\{B^t \cap B^{t'}\}) = 0$ .

In other words, the choice probability measure  $\hat{\pi}$  has no mass where the budget sets intersect. This is convenient because it simplifies some of the definitions that follow.<sup>16</sup>

A random utility is denoted by a measure  $\tilde{\mu}$  over the set of locally nonsatiated utility functions defined on  $\mathbb{R}_+^L$ , which we denote by  $\tilde{\mathcal{U}}$ . The data set  $\tilde{\mathcal{D}}$  is said to be *rationalized by a random utility model* if there exists a random utility  $\tilde{\mu}$  such that for all  $X_t \subset B^t$ ,

$$\hat{\pi}^t(X^t) = \tilde{\mu}(\tilde{\mathcal{U}}(X^t)) \text{ for all } t \in T, \text{ where } \tilde{\mathcal{U}}(X^t) := \left\{ \tilde{U} \in \tilde{\mathcal{U}} : \operatorname{argmax}_{x \in B^t} \tilde{U}(x) \in X^t \right\}.$$

In other words, to rationalize  $\tilde{\mathcal{D}}$  we need to find a distribution on the family of utility functions that generates a demand distribution at each budget set  $B^t$  corresponding to what was observed.

<sup>16</sup>This simplification is not conceptually necessary for the procedure described here or its adaptation to our setting in the next subsection. See the explanations given in [KS](#), all of which also apply here.



Crucially, [McFadden \(2005\)](#) observed that this problem can be discretized as follows. Let  $\{B^{1,t}, \dots, B^{I_t,t}\}$  denote the collection of subsets (which we call *patches*) of the budget  $B^t$  where each subset has as its boundaries the intersection of  $B^t$  with other budget sets and/or the boundary hyperplanes of the positive orthant. Formally, for all  $t \in T$  and  $i_t \neq i'_t$ , each set in  $\{B^{1,t}, \dots, B^{I_t,t}\}$  is closed and convex and, in addition, the following hold:

- (i)  $\cup_{1 \leq i_t \leq I_t} B^{i_t,t} = B^t$ ,
- (ii)  $\text{int}(B^{i_t,t}) \cap B^{t'} = \emptyset$  for all  $t' \neq t$  that satisfy  $B^t \neq B^{t'}$ ,
- (iii)  $B^{i_t,t} \cap B^{i'_t,t} \neq \emptyset$  implies that  $B^{i_t,t} \cap B^{i'_t,t} \subset B^{t'}$  for some  $t' \neq t$  that satisfies  $B^t \neq B^{t'}$ ,

where  $\text{int}(\cdot)$  denotes the relative interior of a set.

We use the vector  $\pi^t \in \Delta^{I_t}$  belonging to the  $I_t$  dimensional simplex to denote the *discretized choice probabilities* over the collection  $\{B^{1,t}, \dots, B^{I_t,t}\}$ . Formally, coordinate  $i_t$  of  $\pi^t$  is given by

$$\pi^{i_t,t} = \hat{\pi}^t(B^{i_t,t}), \quad \text{for all } B^{i_t,t} \in \{B^{1,t}, \dots, B^{I_t,t}\}.$$

Even though there may be  $t, i_t, i'_t$  for which  $B^{i_t,t} \cap B^{i'_t,t} \neq \emptyset$  (as these sets may share parts of their boundaries), [Assumption 1](#) guarantees that  $\pi^t$  is still a probability measure since choice probabilities on the boundaries of  $B^{i_t,t}$  have measure 0. We denote  $\pi := (\pi^1, \dots, \pi^T)'$ .

We call a deterministic data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  *typical* if, for all  $t$ , there is  $i^t$  such that  $x^t \in \text{int}(B^{i^t,t})$  for all  $t \in T$ ; in other words,  $x^t$  lies in the interior of some patch at each observation.<sup>17</sup> If a typical deterministic data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$  satisfies GARP, then for all other  $\hat{x}^t \in \text{int}(B^{i^t,t}), t \in T$ , the data set  $\{(p^t, \hat{x}^t)\}_{t=1}^T$  also satisfies GARP. This is because the revealed preference relations  $\succeq_x, \succ_x$  are determined only by where a choice lies on the budget set relative to its intersection with another budget. Thus, as far as testing rationality is concerned, all choices in a given set  $\text{int}(B^{i^t,t})$  are interchangeable. Therefore, we may classify all typical deterministic data sets according to the patch occupied by the bundle  $x^t$  at each budget set  $B^t$ . In formal terms, we associate to each typical deterministic data set  $\mathcal{D}$  that satisfies GARP a vector  $a = (a^{1,1}, \dots, a^{I_T,T})$  where

$$a^{i_t,t} = \begin{cases} 1 & \text{if } x^t \in B^{i_t,t}, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Notice that we have now partitioned all typical deterministic data sets obeying GARP (of which there are infinitely many) into a *finite* number of distinct classes or types, based on the vector  $a$  associated with each data set. We use  $A$  to denote the matrix whose columns consist of all the  $a$  vectors corresponding to these GARP-obeying types (where

<sup>17</sup>The choice of terminology reflects the the fact that most data sets used in the literature, including ours, are 'typical.'

the columns can be arranged in any order) and use  $|A|$  to denote the number of such columns (which is also the number of types).

The problem of finding a measure  $\tilde{\mu}$  on the family of utility functions to rationalize  $\tilde{\mathcal{D}}$  is essentially one of disaggregating  $\tilde{\mathcal{D}}$  into rationalizable deterministic data sets or, given [Afriat's Theorem](#), into deterministic data sets obeying GARP. If we ignore non-typical deterministic data sets (which is justified because of [Assumption 1](#)), this is in turn equivalent to finding weights on the finitely many distinct types represented by the columns of  $A$ , so that their aggregation at each observation coincides with the discretized choice probabilities. The following result (which we credit to [MR](#) though it does not appear as stated in their work) summarizes these observations.

**MR Theorem.** *Suppose that the stochastic data set  $\tilde{\mathcal{D}} = \{(B^t, \hat{\pi}^t)\}_{t=1}^T$  satisfies [Assumption 1](#). Then  $\tilde{\mathcal{D}}$  is rationalized by a random utility model if and only if there exists a  $v \in \mathbb{R}_+^{|A|}$  such that the discretized choice probabilities  $\pi$  satisfy  $Av = \pi$ .<sup>18</sup>*

Before we turn to the stochastic version of the expenditure-augmented utility model, it is worth highlighting an important shortcoming of the setting envisaged by [MR](#): data sets typically do not take the presumed form  $\tilde{\mathcal{D}} = \{(B^t, \hat{\pi}^t)\}_{t=1}^T$ . This is because even consumers that face the same prices on the  $L$  observed goods (as is commonly assumed in applications with repeated cross sectional data, for instance, [Blundell, Browning, and Crawford \(2008\)](#)) will typically spend different amounts on these goods and, therefore, the observed choices of the population will lie on different budget sets. Furthermore, even if one conditions on that part of the population that chooses the same expenditure level at some vector of prices, as these prices change, they will go on to choose different expenditure levels. The theory set out by [MR](#) to test the random utility model relies heavily on the stylized environment they assume and it is not at all clear how it could be modified to test the model when expenditure is allowed to be endogenous. Recently, [KS](#) operationalized the theory by [MR](#) and developed econometric methodologies to implement it. Moreover, [KS](#) presented an application of their test for rationalizability to the FES data set (one of the two data sets we use in [Section 5](#)), though obtaining  $\{\hat{\pi}^t\}_{t=1}^T$  for this application necessitated estimation of counterfactuals, that is, what the distribution of demand would be if, hypothetically, all consumers were restricted to the same budget set. Inevitably, estimating these distributions will require the use of instrumental variables (to get round the endogeneity of expenditure) and can involve a variety of assumptions on the smoothness of Engel curves, the nature of unobserved heterogeneity across individuals, etc.

<sup>18</sup>Since any  $v$  that satisfies  $Av = \pi$  must also satisfy  $\sum_{j=1}^{|A|} v_j = 1$ , we do not need to explicitly impose the latter condition. While technically trivial, this observation is quite important for the construction of the test.

### 3.2. Rationalization by Random Augmented Utility

Once again, the starting point of our analysis is a stochastic data set  $\mathcal{D} := \{(p^t, \hat{\pi}^t)\}_{t=1}^T$  (of price-probability pairs) which consists of a finite set of distinct prices along with a corresponding distribution over chosen bundles. But there is one important departure from the previous section (where the data consisted of budget-probability pairs): we no longer require the support of  $\hat{\pi}^t$  to lie on the budget set  $B^t$ ; instead the support could be any set in  $\mathbb{R}_+^L$ . In other words, we no longer require all consumers to incur the same expenditure at each price observation; each consumer in the population can decide how much she wishes to spend on the  $L$  observed goods and this could differ across consumers and across price observations. As we pointed out at the end of [Section 3.1](#), this is the form that data typically takes. Also, as in the previous section,  $\hat{\pi}^t$  needs to be estimated in practice and this is the source of statistical uncertainty that we address in [Section 4](#).

A random expenditure-augmented utility is denoted by a measure  $\mu$  over the set of augmented utility functions which we denote by  $\mathcal{U}$ .

**Definition 3.1.** The data set  $\mathcal{D}$  is said to be *rationalized by the random augmented utility model* if there exists a random augmented utility  $\mu$  such that for all  $X_t \subset \mathbb{R}_+^L$ ,

$$\hat{\pi}^t(X^t) = \mu(\mathcal{U}(X^t)) \text{ for all } t \in T, \text{ where } \mathcal{U}(X^t) := \left\{ U \in \mathcal{U} : \operatorname{argmax}_{x \in \mathbb{R}_+^L} U(x, -p^t x) \in X^t \right\}.$$

In actual empirical applications, observations are typically made over time. Therefore, we are effectively asking whether or not  $\mathcal{D}$  is generated by utility maximization over a distribution of augmented utility functions that is stable over the period where observations are taken. This assumption is plausible if (i) there is no change in the prices of the unobserved goods or, more realistically, that these changes could be adequately accounted for by the use of a deflator, and (ii) there is sufficient stability in the way consumers in the population view their long term economic prospects, so that there are only small changes (in the sense of [Proposition 1](#)) in their willingness to trade off consumption of a bundle of goods in  $L$  with the expenditure it incurs.

The problem of finding a measure  $\mu$  to rationalize  $\mathcal{D}$  is essentially one of disaggregating  $\mathcal{D}$  into deterministic data sets rationalizable by augmented utility functions or, given [Theorem 1](#), into deterministic data sets obeying GAPP. Crucially, [Proposition 2](#) tells us that a deterministic data set obeys GAPP if and only if its expenditure-normalized version obeys GARP. It follows that  $\mu$  exists if and only if the normalized version of the stochastic data set  $\mathcal{D}$  obeys the condition identified by MR (as stated in [Section 3.1](#)).

To set this out more formally, we first define the *normalized choice probability*  $\tilde{\pi}^t$  corresponding to  $\hat{\pi}^t$  by scaling observations from the entire orthant onto the budget plane  $B^t$

generated by prices  $p^t$  and expenditure 1. Formally,

$$\tilde{\pi}^t(X^t) = \tilde{\pi}^t \left( \left\{ x : \frac{x}{p^t x} \in X^t \right\} \right), \quad \text{for all } X^t \subset B^t \text{ and all } t \in T.$$

We suppose that [Assumption 1](#) holds on the normalized data set  $\{(B^t, \tilde{\pi}^t)\}_{t=1}^T$ ; abusing the terminology somewhat, we shall say that  $\mathcal{D}$  obeys [Assumption 1](#).<sup>19</sup> We then define the patches on the budgets set  $B^t$  (as in [Section 3.1](#)) and denote them by  $\{B^{1,t}, \dots, B^{I_t,t}\}$ . With these patches in place, we derive the *normalized and discretized choice probabilities*  $\pi^t = (\pi^{1,t}, \dots, \pi^{I_t,t})$  from  $\tilde{\pi}^t$  by assigning to each  $\pi^{i_t,t}$  the normalized choice probability  $\tilde{\pi}^t(B^{i_t,t})$  corresponding to  $B^{i_t,t}$ . Finally, we construct the matrix  $A$ , whose columns are defined by (9); the columns represent distinct GARP-obeying types, which by [Proposition 2](#) coincides with the distinct GAPP-obeying types. The rationalizability of  $\mathcal{D}$  can then be established by checking if there are weights on these types that generate the observed normalized and discretized choice probabilities  $\pi = (\pi^1, \dots, \pi^T)$ . The following result summarizes these observations.

**Theorem 3.** *Let  $\mathcal{D} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$  be a stochastic data set obeying [Assumption 1](#). Then  $\mathcal{D}$  is rationalized by the random augmented utility model if and only if there exists a  $v \in \mathbb{R}_+^{|A|}$  such that the normalized and discretized choice probabilities  $\pi$  (derived from  $\tilde{\pi}$ ) satisfy  $Av = \pi$ .*

It is worth emphasizing that this theorem provides us with a very clean procedure for testing the random augmented utility model. If we were testing the random utility model, then the [MR](#) test would require a data set where expenditures are common across consumers as a starting point; since this is not commonly available it would have to be estimated, which in turn requires an additional econometric procedure with all its attendant assumptions. By contrast, to test the random augmented utility model, all we have to do is apply the [MR](#) test to the expenditure-normalized data set  $\tilde{\mathcal{D}} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$ , which is obtained (via a simple transformation purely as a consequence of the theoretical model itself) from the original data set  $\mathcal{D} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$ .

We end this subsection with an example that makes explicit the operationalization of [Theorem 3](#) using data where the normalized and discretized choice probabilities are determined by the sample frequency of choices.

**Example 3.** Suppose the econometrician observes the set of ten choices at two price vectors,  $p^t = (2, 1)$  and  $p^{t'} = (1, 2)$ , given by the black points in [Figures 4a](#) and [4b](#). These

<sup>19</sup>A sufficient condition for [Assumption 1](#) is that  $\tilde{\pi}$  assigns 0 probability to sets with a Lebesgue measure of 0. Also, as before, this is merely for ease of exposition: our test does not depend on this assumption and, importantly, the data in our empirical application satisfies [Assumption 1](#).

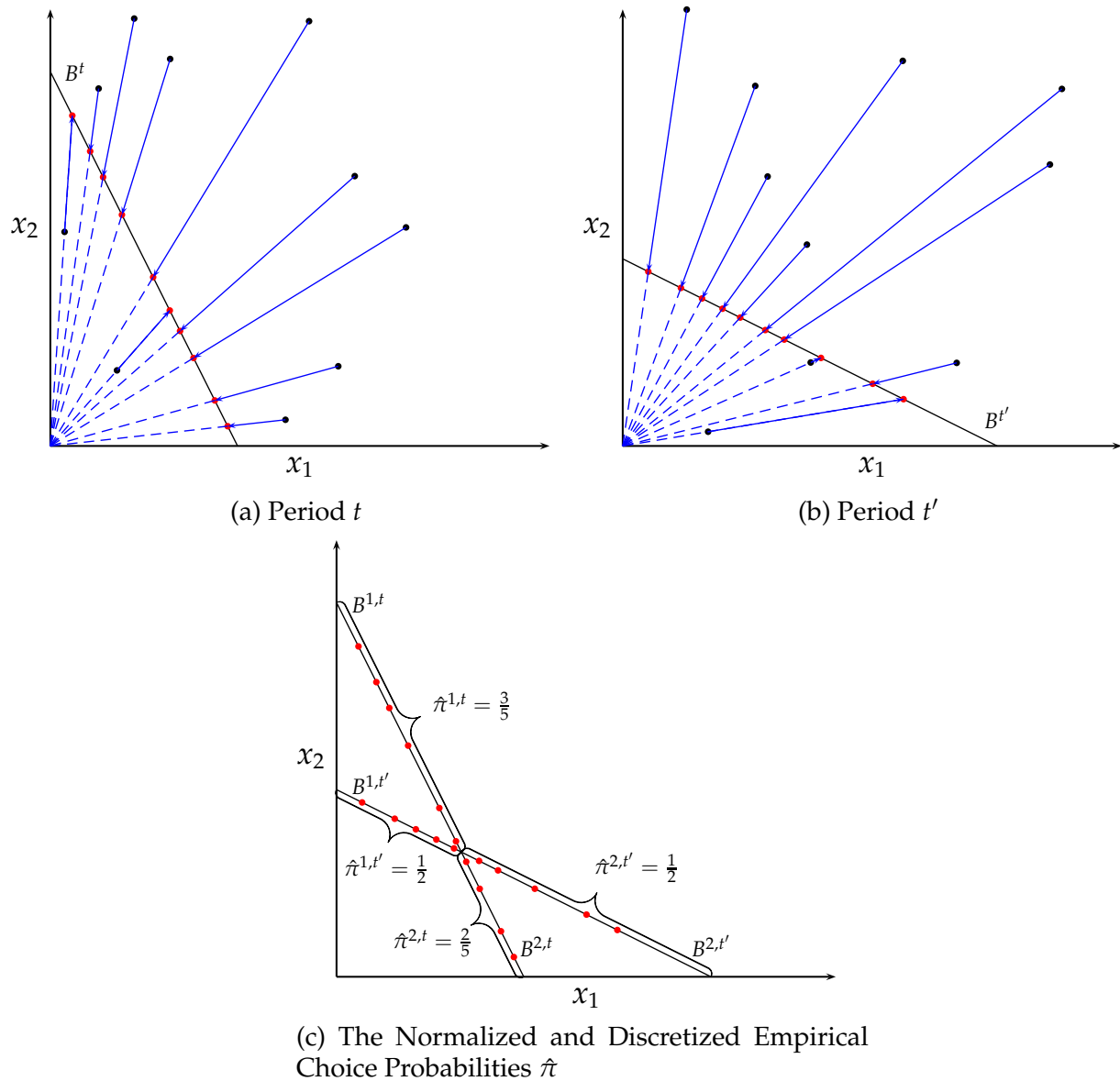


FIGURE 4. Observed and Scaled Choice Data

figures also demonstrate how the choices are scaled (to the red points) on to the normalized budget sets. Figure 4c then shows that the choice probabilities

$$\hat{\pi} = \left( \frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2} \right)'$$

(the hat notation refers to the fact that the choice probabilities are derived from the sample choice frequencies which is how we estimate choice probabilities) are determined by the proportion of the normalized choices that lie on each segment of the budget lines. Lastly, Figure 5 illustrates the various rational types for these two budget sets. Note that GARP (equivalently, GAPP) violations only occur when the choices lie on  $B^{2,t}$  and  $B^{1,t'}$ . The

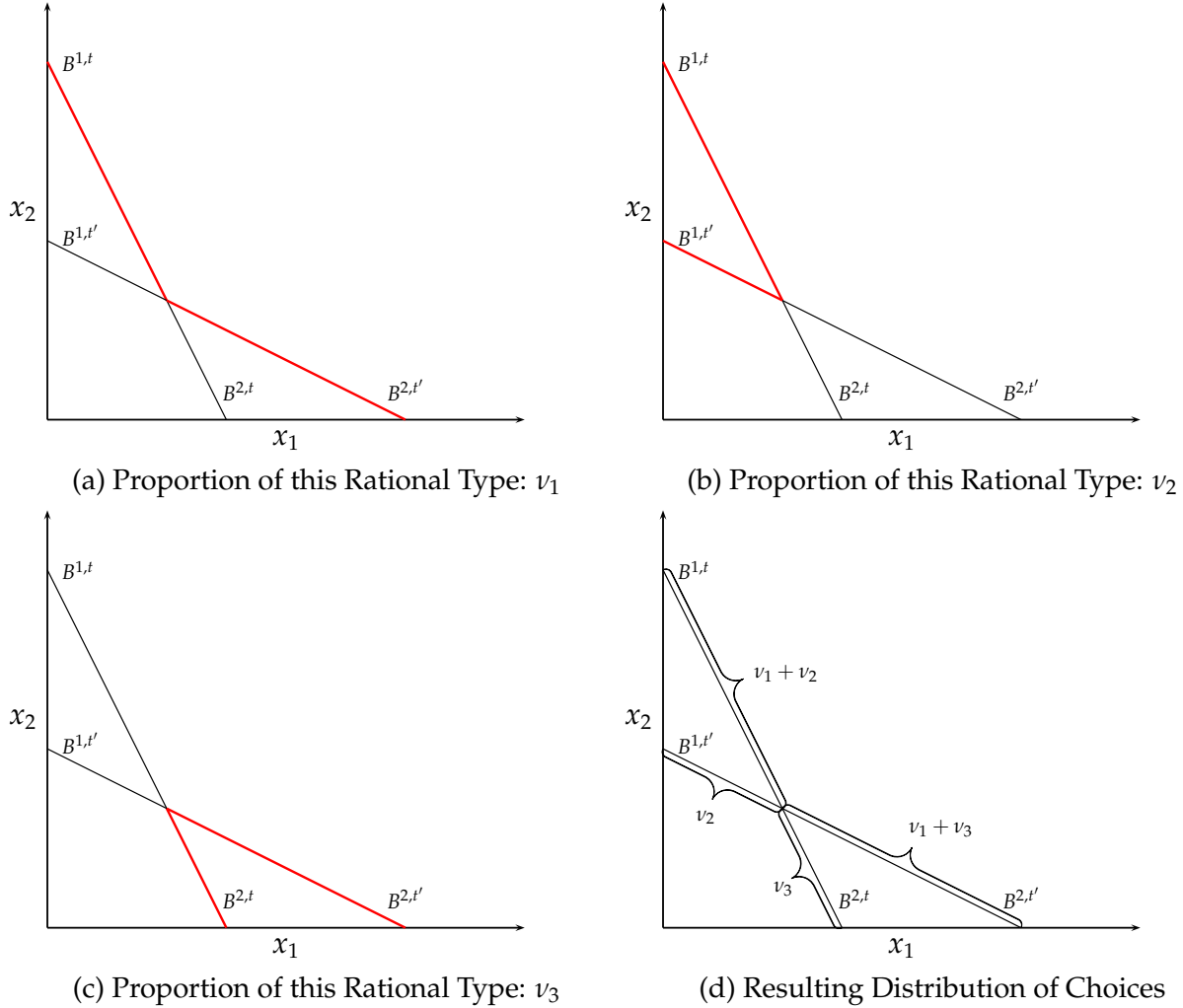


FIGURE 5. Distribution of Rational Types

resulting  $A$  matrix is given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and, additionally, } Av = \begin{pmatrix} \nu_1 + \nu_2 \\ \nu_3 \\ \nu_2 \\ \nu_1 + \nu_3 \end{pmatrix}.$$

The first, second and third column of  $A$  correspond to the three types of GARP-consistent demand behavior, which are depicted in Figures 5a, 5b and 5c respectively. If the proportion of the three types in the population is  $\nu_1$ ,  $\nu_2$  and  $\nu_3$ , the resulting distribution on the segments of the budget sets are given by  $Av$ , the expression for which is displayed above and depicted in Figure 5d. Theorem 3 says that rationalization is equivalent to the existence of  $\nu \in \mathbb{R}_+^3$  such that  $Av = \hat{\pi}$ .

The data from this example can be rationalized by the distribution of rational types

$$\nu = \left( \frac{1}{10}, \frac{1}{2}, \frac{2}{5} \right)'.$$

While the rationalizing distribution  $\nu$  is unique in this example, this is typically not the case (as the equation  $A\nu = \pi$  may have multiple solutions for  $\nu$ ). Notice also that it is not the case that a solution always exists. Indeed, if  $\hat{\pi}^{1,t'} > \hat{\pi}^{1,t}$ , then the choice probabilities would not be rationalizable as  $\nu_2 > \nu_1 + \nu_2$  is not possible.

### 3.3. Welfare Comparisons

Since the test for rationalizability involves finding a distribution  $\nu$  over different types, it is possible to use this distribution for welfare comparisons: for any two prices in the data set and given a distribution  $\nu$  that rationalizes  $\mathcal{D}$ , we can determine the proportion of types who are revealed better off and the proportion who are revealed worse off. However, since there may be multiple  $\nu$  that satisfy  $A\nu = \pi$ , the welfare rankings extractable from the data will typically be in terms of bounds.

To be specific, suppose we would like to determine the welfare effect of a price change from  $p^{t'}$  to  $p^t$ . Let  $\mathbb{1}_{t \succeq_p^* p^{t'}}$  denote a vector of length  $|A|$  such that the  $j^{\text{th}}$  element is 1 if  $p^t \succeq_p^* p^{t'}$  for the rational type corresponding to column  $j$  of  $A$  and 0 otherwise. In words,  $\mathbb{1}_{t \succeq_p^* p^{t'}}$  enumerates the set of rational types for which  $p^t$  is revealed preferred to  $p^{t'}$ . For a rationalizable data set  $\mathcal{D}$ , the solution to the optimization problem

$$\underline{\mathcal{N}}_{t \succeq_p^* p^{t'}} := \begin{array}{ll} \min_{\nu} & \mathbb{1}_{t \succeq_p^* p^{t'}} \nu, \\ \text{subject to} & A\nu = \pi, \end{array} \quad (10)$$

gives the lower bound, while

$$\overline{\mathcal{N}}_{t \succeq_p^* p^{t'}} := \begin{array}{ll} \max_{\nu} & \mathbb{1}_{t \succeq_p^* p^{t'}} \nu, \\ \text{subject to} & A\nu = \pi, \end{array} \quad (11)$$

gives the upper bound on the proportion of consumers who are revealed better off at prices  $p^t$  compared to  $p^{t'}$ .

Since (10) and (11) are both linear programming problems (which have solutions if, and only if,  $\mathcal{D}$  is rationalizable), they are easy to implement and computationally efficient. Suppose that the solutions are  $\underline{\nu}$  and  $\overline{\nu}$  respectively; then for any  $\beta \in [0, 1]$ ,  $\beta\overline{\nu} + (1 - \beta)\underline{\nu}$  is also a solution to  $A\nu = \pi$  and, in this case, the proportion of consumers who are revealed better off at  $p^t$  compared to  $p^{t'}$  is exactly  $\beta \underline{\mathcal{N}}_{t \succeq_p^* p^{t'}} + (1 - \beta) \overline{\mathcal{N}}_{t \succeq_p^* p^{t'}}$ . In other words, the proportion of consumers who are revealed better off can take any value in the interval  $[\underline{\mathcal{N}}_{t \succeq_p^* p^{t'}}, \overline{\mathcal{N}}_{t \succeq_p^* p^{t'}}]$ .

**Proposition 3** tells us that the revealed preference relations are tight, in the sense that if, for some consumer,  $p^t$  is not revealed preferred to  $p^{t'}$  then there exists an augmented

utility function which rationalizes her consumption choices and for which she strictly prefers  $p^{t'}$  to  $p^t$ . Given this, we know that, amongst all rationalizations of  $\mathcal{D}$ ,  $\underline{N}_{t \succeq_{p^{t'}}}$  is also the infimum on the proportion of consumers who are better off at  $p^t$  compared to  $p^{t'}$ . At the other extreme, we know that there is a rationalization for which the proportion of consumers who are revealed better off at  $p^{t'}$  compared to  $p^t$  is as low as  $\underline{N}_{t' \succeq_{p^t}}$ .<sup>20</sup> Applying [Proposition 3](#) again, a rationalization could be chosen such that all other consumers prefer  $p^t$  to  $p^{t'}$ . Therefore, across all rationalizations of  $\mathcal{D}$ ,  $1 - \underline{N}_{t' \succeq_{p^t}}$  is the supremum on the proportion of consumers who prefer  $p^t$  and  $p^{t'}$ . Lastly, since the set of distributions  $\nu$  that rationalize  $\mathcal{D}$  form a convex set, the true proportion of consumers who are better off can lie anywhere between the two extremes identified.

The following proposition summarizes these observations.

**Proposition 4.** *Let  $\mathcal{D} = \{(p^t, \pi^t)\}_{t=1}^T$  be a stochastic data set that satisfies [Assumption 1](#) and is rationalized by the augmented utility model.*

- (1) *Then for every  $\eta \in [\underline{N}_{t \succeq_{p^{t'}}}, \overline{N}_{t \succeq_{p^{t'}}}]$ , there is rationalization of  $\mathcal{D}$  for which  $\eta$  is the proportion of consumers who are revealed better off at  $p^t$  compared to  $p^{t'}$ .*
- (2) *For any rationalization of  $\mathcal{D}$ , there is a proportion of consumers who are better off at  $p^t$  compared to  $p^{t'}$ . This proportion can take any value in the interval  $[\underline{N}_{t \succeq_{p^{t'}}}, 1 - \underline{N}_{t' \succeq_{p^t}}]$ .*

It may be helpful to consider how [Proposition 4](#) applies to [Example 3](#). In that case, there is a unique solution to  $Av = \pi$  and the proportion of consumers who are revealed better off at  $p^t$  compared to  $p^{t'}$  is  $\nu_2 = 1/2$ , while the proportion who are revealed better off at  $p^{t'}$  to  $p^t$  is  $\nu_3 = 2/5$ . Those consumers who belong to neither of these two types could be either better or worse at  $p^t$  compared to  $p^{t'}$ . Therefore, across all rationalizations of that data set, the proportion of consumers who are better off at  $p^t$  compared to  $p^{t'}$  can be as low as  $1/2$  and as high as  $1 - 2/5 = 3/5$ .

#### 4. THE ECONOMETRIC METHODOLOGY

In this section, we develop the econometric methodology required to test our model and conduct welfare analysis. Specifically, we first briefly discuss how [Theorem 3](#) allows us to statistically test the random augmented utility model by using the methodology developed by [KS](#). We then present the novel econometric theory that is required to extract the welfare bounds (characterized in statement (1) of [Proposition 4](#)) from the data. As shall become clear, the procedure we develop (for inference on linear transforms of partially identified vectors) has other applications beyond our specific empirical analysis; specifically, we provide new algorithms for the calculation and statistical inference in nonparametric counterfactual analysis for a general partially identified model.

<sup>20</sup>Because of [Assumption 1](#), we could assume that this is a strict revealed preference.



#### 4.1. Testing the Random Augmented Utility Model

Recall from [Theorem 3](#) that, given a stochastic data set  $\mathcal{D} = \{(p^t, \hat{\pi}^t)\}_{t=1}^T$ , a test of the random augmented utility model is a test of the hypothesis

$$H_0 : \exists v \in \mathbb{R}_+^{|A|} \text{ such that } Av = \pi,$$

where  $\pi$  is the normalized and discretized choice probability vector derived from  $\hat{\pi}$ . [KS](#) restate this test in an equivalent (and more convenient) form

$$H_0 : \min_{v \in \mathbb{R}_+^{|A|}} [\pi - Av]' \Omega [\pi - Av] = 0, \quad (12)$$

where  $\Omega$  is a positive definite matrix. The solution of the above minimization problem is the projection of  $\pi$  onto the cone  $\{Av \mid v \in \mathbb{R}_+^{|A|}\}$  under the weighted norm  $\|\zeta\|_\Omega = \sqrt{\zeta' \Omega \zeta}$ , where  $\zeta \in \mathbb{R}^{|A|}$ . The corresponding value of the objective function is the squared length of the projection residual vector and is zero if, and only if,  $\pi$  can be stochastically rationalized.

Of course, in practice, the data corresponding to each price  $p^t$  is not in the form of a distribution  $\hat{\pi}^t$  but instead consists of a cross section of choices  $\{x_{n_t}^t\}_{n_t=1}^{N_t}$  made by  $N_t$  households. We estimate  $\pi$  by its sample analog  $\hat{\pi} = (\hat{\pi}^1, \dots, \hat{\pi}^T)$  where each  $\hat{\pi}^t$  is obtained by normalizing and discretizing the empirical distribution of the observed choices  $\{x_{n_t}^t\}_{n_t=1}^{N_t}$  (as in [Figure 4](#)).<sup>21</sup> To test (12), [KS](#) use a sample counterpart

$$J_N := N \min_{v \in \mathbb{R}_+^{|A|}} [\hat{\pi} - Av]' \Omega [\hat{\pi} - Av], \quad (13)$$

where  $N = \sum_{t=1}^T N_t$  denotes the total number of observations (the sum of the number of households across years). Here, the minimizing value  $\hat{v}$  may not be unique but the implied choice probabilities  $\hat{\eta} := A\hat{v}$  are. Note that  $\hat{\eta} = \hat{\pi}$  and  $J_N = 0$  iff the sample choice frequencies can be rationalized by a random augmented utility model. In this case, the null hypothesis is trivially accepted. We determine critical values for the test statistic by employing the modified bootstrap procedure of [KS](#).

#### 4.2. Nonparametric inference for counterfactuals

Recall that our model allows for welfare comparisons where, for any two prices  $p^t$  and  $p^{t'}$  in our data, we can determine bounds  $[\underline{N}_{t \succeq p^{t'}}, \overline{N}_{t \succeq p^{t'}}]$  on the proportion of the population that is better off at the former prices. This is an instance of a more general problem of determining bounds on a counterfactual object given by a linear transform

$$\theta = \rho v \quad \text{subject to } Av = \pi,$$

<sup>21</sup>In the data we employ for our empirical application, there are no observations that lie on the intersection of any two normalized budget sets so  $\hat{\pi}^t$  determines a probability measure on  $\{B^{1,t}, \dots, B^{I,t}\}$ . In other words, the data in our empirical application satisfies [Assumption 1](#).

where  $\rho \in \mathbb{R}^{|A|}$  is a given vector and  $\theta \in \mathbb{R}$  is the parameter of interest. In our main application,  $\rho = \mathbb{1}_{t \succeq_p^* p^t}$  (which, recall, is a vector where the  $j^{\text{th}}$  element is 1 if  $p^t \succeq_p^* p^t$  for the rational type corresponding to column  $j$  of  $A$  and 0 otherwise) and  $\theta = \mathcal{N}_{t \succeq_p^* p^t}$ . It is worth stressing that the methodology we develop here is valid not just for our model, but for inference in any setting where identification has the same structure (such as [KS](#)). In this section, we provide a high-level description of our testing procedure, the formal treatment can be found in [Appendix B](#).

The goal is to calculate and do inference for the set of counterfactuals  $\Theta_I$ , which is given by

$$\Theta_I := \{\theta \mid \pi \in \mathcal{S}(\theta)\}, \text{ where } \mathcal{S}(\theta) := \left\{ Av \mid \rho v = \theta, v \in \Delta^{|A|-1} \right\}.$$

We do this by inverting a test of

$$\pi \in \mathcal{S}(\theta) \tag{14}$$

or, equivalently,

$$\min_{v \in \Delta^{|A|-1}, \theta = \rho v} [\pi - Av]' \Omega [\pi - Av] = 0.$$

We use a sample counterpart of the above display as the test statistic:

$$\begin{aligned} J_N(\theta) &= \min_{v \in \Delta^{|A|-1}, \theta = \rho v} [\hat{\pi} - Av]' \Omega [\hat{\pi} - Av] \\ &= \min_{\eta \in \mathcal{S}(\theta)} N[\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta]. \end{aligned}$$

The challenge is to compute an appropriate critical value for this test statistic which accounts for the fact that its limiting distribution depends discontinuously on nuisance parameters in a very complex manner.<sup>22</sup> This issue is similar to that faced in the canonical moment inequality testing problem (see, for instance, [Andrews and Soares \(2010\)](#)) which has a tight theoretical link, and is in fact the dual, to our problem. Formally, it can be shown that  $\pi \in \mathcal{S}(\theta)$  if, and only if,

$$\pi \text{ satisfies } B\pi \leq 0, \tilde{B}\pi = d(\theta) \text{ and } \mathbf{1}'\pi = 1, \tag{15}$$

for some fixed matrices  $B, \tilde{B}$  and a vector  $d$  that depends on  $\theta$  (see the [Appendix B](#) for details), where  $\mathbf{1}$  is the  $|A|$ -vector of ones. Importantly, for fixed  $\theta$  and prices  $\{p_t\}_{t=1}^T$ , these objects are known to exist and are nonstochastic.

That said, the reason we cannot directly use testing procedures from the moment inequality literature (although we do draw on this literature for the proof of the asymptotic validity of our method) is that numerically deriving the dual representation (15) is computationally prohibitive for problems of empirically relevant scale (such as our empirical

<sup>22</sup>It is, in principle, possible to derive critical values by  $(m < n-)$ subsampling. While asymptotically valid, critical values derived in this way can be quite conservative, as has been forcefully argued in [Andrews and Soares \(2010\)](#), page 137.

application). This computational bottleneck requires us to work directly with (14) and so we instead follow a *tightening* procedure.

The tightening procedure we employ is similar in spirit to that used in KS but has to account for the additional features of our testing problem. In particular, establishing the validity of the tightening method is more intricate as it depends on certain geometric properties of the set  $\mathcal{S}(\theta)$ . Moreover, unlike the hypothesis considered in KS, our hypothesis (14) depends on a parameter  $\theta$  that lives in a compact space  $\Theta := \{\rho\nu : \nu \in \Delta^{|A|-1}\}$  and may in practice be close to the boundary of that space. This requires a delicate *restriction-dependent* tightening, which now we describe.

Let  $\tau_N$  be a tuning parameter chosen such that  $\tau_N \downarrow 0$  and  $\sqrt{N}\tau_N \uparrow \infty$  and, for  $\theta \in \Theta_I$ , define the  $\tau_N$ -tightened version  $\mathcal{S}_{\tau_N}(\theta)$  of  $\mathcal{S}(\theta)$ , that is,

$$\mathcal{S}_{\tau_N}(\theta) := \{Av \mid \rho\nu = \theta, \nu \in \mathcal{V}_{\tau_N}(\theta)\},$$

where  $\mathcal{V}_{\tau_N}(\theta)$  is obtained by appropriately constricting (tightening) the set  $\Delta^{|A|-1}$ , the parameter space for  $\nu$  in the definition of  $\mathcal{S}$ . (Appendix B contains the formal definition of  $\mathcal{V}_{\tau_N}(\theta)$ ). As the notation suggests, the way we tighten  $\mathcal{V}_{\tau_N}(\theta)$  is dependent on the hypothesized value  $\theta$  in order to deal with the aforementioned issue associated with the boundary of the parameter space  $\Theta$ . Note that  $\mathcal{V}_{\tau_N}(\theta)$  converges to  $\Delta^{|A|-1}$ , hence  $\mathcal{S}_{\tau_N}(\theta)$  to  $\mathcal{S}$  as  $\tau_N \rightarrow 0$ , for every value of  $\theta$ . Though the definition of  $\mathcal{V}_{\tau_N}(\theta)$  provided in Appendix B for a general (linear) counterfactual is rather involved, we show that it simplifies for the welfare comparison application considered in this paper.

The set  $\mathcal{S}_{\tau_N}(\theta)$  replaces  $\mathcal{S}(\theta)$  in the bootstrap population, inducing locally conservative distortions, which guard against discontinuities in the test statistic's limiting distribution as a function of nuisance parameters. In Appendix B, we show that tightening procedure turns inequality constraints into binding ones in the bootstrap DGP if their degrees of slackness are small, though inequalities with large degrees of slackness remain non-binding after tightening. This prevents the bootstrap procedure from over-rejection when the sampling DGP is near the boundary of inequality constraints implied by (14).

Our bootstrap algorithm proceeds as follows. For each  $\theta \in \Theta$

- (i) Compute the  $\tau_N$ -tightened restricted estimator of the empirical choice distribution

$$\hat{\eta}_{\tau_N} := \underset{\eta \in \mathcal{S}_{\tau_N}(\theta)}{\operatorname{argmin}} N[\hat{\pi} - \eta]' \Omega[\hat{\pi} - \eta].$$

- (ii) Define the  $\tau_N$ -tightened recentered bootstrap estimators

$$\hat{\pi}_{\tau_N}^{*(r)} := \hat{\pi}^{*(r)} - \hat{\pi} + \hat{\eta}_{\tau_N}, \quad r = 1, \dots, R,$$

where  $\hat{\pi}^{*(r)}$  is a straightforward bootstrap analog of  $\hat{\pi}$  and  $R$  is the number of bootstrap samples. For instance, in our application,  $\hat{\pi}^{*(r)}$  is generated by the simple nonparametric bootstrap of choice frequencies. This is the step that makes constraints with small slack binding.

(iii) For each  $r = 1, \dots, R$ , compute the value of the bootstrap test statistic

$$J_{N, \tau_N}^{*(r)}(\theta) = \min_{\eta \in \mathcal{S}_{\tau_N}(\theta)} N[\hat{\pi}_{\tau_N}^{*(r)} - \eta]' \Omega [\hat{\pi}_{\tau_N}^{*(r)} - \eta].$$

(iv) Use the empirical distribution of  $J_{N, \tau_N}^{*(r)}(\theta), r = 1, \dots, R$  to obtain the critical value for  $J_N(\theta)$ .

We obtain a confidence interval for  $\theta$  by collecting the values of  $\theta$  that are not rejected by the bootstrap implemented with the algorithm (i)-(iv).

The theorem below establishes the asymptotic validity of the above procedure. Let

$$\mathcal{F} := \{(\theta, \pi) \mid \theta \in \Theta, \pi \in \mathcal{S}(\theta) \cup \mathcal{P}\}$$

where  $\mathcal{P}$  denote the set of all  $\pi$ 's that satisfy Condition 1 in Appendix for some (common) value of  $(c_1, c_2)$ .

**Theorem 4.** Choose  $\tau_N$  so that  $\tau_N \downarrow 0$  and  $\sqrt{N}\tau_N \uparrow \infty$ . Also, let  $\Omega$  be diagonal, where all the diagonal elements are positive. Then under Assumptions 2 and 3,

$$\liminf_{N \rightarrow \infty} \inf_{(\theta, \pi) \in \mathcal{F}} \Pr\{J_N(\theta) \leq \hat{c}_{1-\alpha}\} = 1 - \alpha,$$

where  $0 \leq \alpha \leq \frac{1}{2}$  and  $\hat{c}_{1-\alpha}$  is the  $1 - \alpha$  quantile of  $J_{N, \tau_N}^*$ .

## 5. EMPIRICAL APPLICATION

In this section, we test our model and conduct welfare analysis on two separate data sets: the U.K. Family Expenditure Survey (FES) and the Canadian Surveys of Household Spending (SHS). The aim of the empirical analysis is to show that the data supports the model and to demonstrate that the estimated welfare bounds are informatively tight.

We first present the analysis for the FES as it is widely used in the nonparametric demand estimation literature (for instance, by [Blundell, Browning, and Crawford \(2008\)](#), [KS, Hoderlein and Stoye \(2014\)](#), [Adams \(2016\)](#), [Kawaguchi \(2017\)](#)). In the FES, about 7000 households are interviewed each year and they report their consumption expenditures in a variety of different commodity groups. In addition, the data contains a variety of other demographic information (most of which we do not need nor exploit). Consumption bundles are derived from the expenditures by using the annual (real) price indices for these commodity groups taken from the annual Retail Prices Index. To directly contrast our analysis with that of [Blundell, Browning, and Crawford \(2008\)](#), we follow them

and restrict attention to households with cars and children, leaving us with roughly 25% of the original data.

We implement tests for 3, 4, and 5 composite goods. The coarsest partition of 3 goods—food, services, and nondurables—is precisely what is examined by [Blundell, Browning, and Crawford \(2008\)](#) (and we use their replication files). As in [KS](#), we increase the dimension of commodity space by first separating out clothing and then alcoholic beverages from the nondurables. Recall that the test of our model involves projecting all the consumption bundles for a given year onto a single budget set corresponding to expenditure 1 (as in [Figure 5d](#)), and choice probabilities are then estimated by the corresponding sample frequencies. It is worth reiterating this strength of our model; since our test can be directly applied to the data, it avoids the layers of smoothing, endogeneity adjustment, and, in some cases, aggregation conditions that characterize other analyses of these data.

While it is, in principle, possible to conduct a single test on the entire data from 1975 to 1999, we instead implement the test in blocks of 6 years. We do so mainly for two reasons. The first is practical; the test involves finding a distribution ( $\nu$ ) over the set of possible rational types and the computational complexity of the test depends the size of this set (given by the size of the  $A$  matrix). This in turn is determined by the number of distinct patches formed by the intersection of the budget sets and this grows exponentially in the number of years.<sup>23</sup> The second reason is that the main assumption of the model—a time-invariant distribution of augmented utility functions—is more plausible over shorter time horizons. Indeed, we might not expect this to hold over the entire time horizon spanned by the data because there are first-order changes to the U.K. income distribution over this period ([Jenkins, 2016](#)).

[Table 1](#) (columns correspond to different blocks of 6 years and rows contain the values of the test statistic and the corresponding p-values) shows that our model is not rejected by the FES data.<sup>24</sup> A similar result is also obtained by [KS](#) (the [MR](#) model is not rejected on these data either) but p-values are hard to compare because their analysis incurs two additional layers of statistical noise by smoothing over expenditure (done by series estimation) and by adjusting for endogeneity. A consequence of this additional noise is that (ceteris paribus) the test of our model is expected to have higher power, as it has nontrivial power against deviations of the order  $N^{-1/2}$  from the null, whereas smoothing-based tests requires deviations of the order of a nonparametric rate to have nontrivial power.

We also estimated the bounds  $[\underline{N}_{t \geq p^t}, \overline{N}_{t \geq p^t}]$  on the proportion of households that are better off at prices  $p^t$  than at prices  $p^{t'}$ . For brevity, we present a few representative estimates using data for the years 1975-1980 in [Table 2](#). The column ‘Estimated Bounds’

<sup>23</sup>This computational constraint prevents us from testing more than 8 years at a time.

<sup>24</sup>The proof of the asymptotic validity of the test of our model presumes that  $\alpha \leq 0.5$  and so p-values above 0.5 should not be interpreted to indicate anything other than nonsignificance at interesting test sizes.

		Year Blocks									
		75-80	76-81	77-82	78-83	79-84	80-85	81-86	82-87	83-88	84-89
3 Goods	Test Statistic ( $J_N$ )	0.337	0.917	0.899	0.522	0.018	0.082	0.088	0.095	0.481	0.556
	p-value	0.04	0.34	0.55	0.59	0.99	0.67	0.81	0.91	0.61	0.48
4 Goods	Test Statistic ( $J_N$ )	0.4	0.698	0.651	0.236	0.056	0.036	0.037	0.043	0.043	0.232
	p-value	0.25	0.58	0.63	0.91	0.96	0.99	0.96	0.95	0.99	0.68
5 Goods	Test Statistic ( $J_N$ )	0.4	0.687	0.705	0.329	0.003	0.082	0.088	0.104	0.103	0.144
	p-value	0.3	0.66	0.68	0.88	0.999	0.96	0.79	0.85	0.9	0.83

		Year Blocks									
		85-90	86-91	87-92	88-93	89-94	90-95	91-96	92-97	93-98	94-99
3 Goods	Test Statistic ( $J_N$ )	0.027	1.42	2.94	1.51	1.72	0	0.313	0.7	0.676	0.26
	p-value	0.69	0.3	0.18	0.24	0.21	1	0.59	0.48	0.6	0.83
4 Goods	Test Statistic ( $J_N$ )	0.227	0.025	0.157	0.154	0.004	1.01	0.802	0.872	0.904	0.604
	p-value	0.48	0.96	0.8	0.73	0.97	0.21	0.31	0.57	0.65	0.74
5 Goods	Test Statistic ( $J_N$ )	0.031	0.019	0.018	0.019	0.023	0.734	0.612	0.643	0.634	0.488
	p-value	0.85	0.98	0.97	0.91	0.83	0.22	0.4	0.72	0.78	0.79

TABLE 1. Test Statistics and p-values for sequences of 6 budgets of the FES. Bootstrap size is  $R = 1000$ .

Comparison	Estimated Bounds	Confidence Interval
$p^{1976} \succ_p^* p^{1977}$	[.150, .155]	[.13, .183]
$p^{1977} \succ_p^* p^{1976}$	{.803}	[.784, .831]
$p^{1979} \succ_p^* p^{1980}$	[.517, .530]	[.487, .56]
$p^{1980} \succ_p^* p^{1979}$	{.463}	[.436, .497]

TABLE 2. Estimated bounds and confidence intervals for the proportion of consumers who reveal prefer one price to another one in the FES data. Data used are for 1975-1980. Bootstrap size is  $R = 1000$ .

are the bounds obtained by calculating  $\mathbb{1}_{t \succeq_p^* t'} \nu$  from the (not necessarily unique) values of  $\nu$  that minimize the test statistic (13). Note that in two cases this estimate is unique while it is not in the other two cases. In either scenario, the procedure set out for calculating confidence intervals in Section 4.2 is applicable and they give the intervals displayed (which must necessarily contain the estimated bounds). It is worth noting that the width of these intervals is less than .1 throughout, so they are quite informative.

For our second empirical application using Canadian data, we use the replication kit of Norris and Pendakur (2013, 2015). Like the FES, the SHS is a publicly available, annual data set of household expenditures in a variety of different categories. It also contains rich

Province		Year Blocks							
		97-02	98-03	99-04	00-05	01-06	02-07	03-08	04-09
Alberta	Test Statistic ( $J_N$ )	.07	0	0	0	0	0	.003	4.65
	p-value	.94	1	1	1	1	1	.98	.04
British Columbia	Test Statistic ( $J_N$ )	.89	.56	.48	.07	.05	6.23	8.87	8.71
	p-value	.47	.47	.98	.96	.97	.05	.02	.01
Manitoba	Test Statistic ( $J_N$ )	0	0	0	0	0	0	.01	.01
	p-value	1	1	1	1	1	1	1	1
New Brunswick	Test Statistic ( $J_N$ )	.08	.05	0	0	0	.60	.58	.57
	p-value	.94	.94	1	1	1	.58	.79	.68
Newfoundland	Test Statistic ( $J_N$ )	.10	.32	.29	.29	.38	3.08	2.30	2.08
	p-value	.85	.90	.91	.87	.81	.21	.35	.27
Nova Scotia	Test Statistic ( $J_N$ )	.05	.03	0	0	0	0	.93	1.02
	p-value	.97	.98	1	1	1	1	.69	.58
Ontario	Test Statistic ( $J_N$ )	.064	.040	.035	0	0	0	0	0
	p-value	.98	.95	.91	1	1	1	1	1
Quebec	Test Statistic ( $J_N$ )	.11	0	0	0	0	.51	.54	.49
	p-value	.88	1	1	1	1	.67	.67	.65
Saskatchewan	Test Statistic ( $J_N$ )	0	0	0	0	0	.02	.02	0
	p-value	1	1	1	1	1	1	1	1

TABLE 3. Test Statistics and p-values for sequences of 6 budgets of the SHS. Bootstrap size is  $R = 1000$ .

demographic data (most of which we once again do not need nor use). We study annual expenditure in 5 categories that constitute a large share of the overall expenditure on non-durables: food purchased (at home and in restaurants), clothing and footwear, health and personal care, recreation, and alcohol and tobacco. The SHS data is rich enough to allow us to analyze the data separately for the nine most populous provinces: Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland, Nova Scotia, Ontario, Quebec, and Saskatchewan. The number of households in each province-year range from 291 (Manitoba, 1997) to 2515 (Ontario, 1997). We use province-year prices indices (as constructed by [Norris and Pendakur \(2015\)](#)) and deflate them using province-year CPI data from Statistics Canada to get real price indices.

Table 4 displays the test statistics and associated p-value for each province and every 6 year block. Compared to the FES data, there are two notable differences. The first is that many more test statistics are exactly zero; that is, the observed choice frequencies are rationalized by the random augmented utility model. The second is that, for a small

Comparison	Estimated Bounds	Confidence Interval
$p^{1998} \succ_p^* p^{2001}$	{.099}	[.073, .125]
$p^{2001} \succ_p^* p^{1998}$	{.901}	[.875, .927]
$p^{1999} \succ_p^* p^{2002}$	[.299, .341]	[.272, .385]
$p^{2002} \succ_p^* p^{1999}$	[.624, .701]	[.594, .728]

TABLE 4. Estimated bounds and confidence intervals for the proportion of consumers who reveal prefer one price to another one in the SHS data. Data used are for 1997-2002 in British Columbia. Bootstrap size is  $R = 1000$ .

proportion of year blocks, there are statistically significant positive test statistics (see, for instance, the last three columns for British Columbia). These low p-values suggest that the conservative distortion which we incur to guarantee uniform validity is modest. Additionally, the p-values taken together do not reject the model if multiple testing is taken into account; for example, step-down procedures would terminate at the first step (that is, Bonferroni adjustment). Finally, we also provide some sample welfare bounds in Table 4. The bounds for the SHS are qualitatively similar to those computed for the FES: there are instances of point identification and in the case of partial identification, the bounds are informative and no wider than .1.

We end this section with two remarks. First, while there is an indirect effect through the potential size of the matrix  $A$ , neither the computational complexity of our test nor its statistical power depend directly on the dimension of commodity space. In particular, we avoid a statistical curse of dimensionality and most of our tests for a five dimensional commodity space can be replicated on everyday equipment. Second, we note that finite sample power does not necessarily increase in the number of years being tested or the dimension of the commodity space. Either could increase the set of revealed preference relations to be tested which will increase power if one of them is in fact violated; but this has to be traded off against increased noise.

## 6. CONCLUSION

We developed a revealed preference analysis of a model of consumption in which the consumer maximizes utility over an observed set of purchases, taking into account a disutility of expenditure, but is not subjected to a hard budget constraint. Our model naturally generalizes to a random utility context which is suitable for demand analysis using repeated cross-section data. We show how to statistically test the model and develop novel econometric theory for inference on the proportion of the population that benefits from a price change (and, more generally, for determining bounds on linear transforms of partially identified vectors). The model is supported by empirical evidence from Canadian



and U.K. data and these applications show that meaningful welfare comparisons can be extracted from the data.

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## APPENDIX A. OMITTED PROOFS FROM SECTION 2

**PROOF OF PROPOSITION 1.** Our proof of part (1) relies on a result of Varian (1988), which says that given any data set  $\mathcal{D}$ , there always exist  $\{z^t\}_{t=1}^T$  such that the augmented data set  $\{((p^t, 1), (x^t, z^t))\}_{t=1}^T$  obeys GARP. By Afriat's Theorem, there is a utility function  $\tilde{U} : \mathbb{R}^{L+1} \rightarrow \mathbb{R}$  such that  $(x^t, z^t)$  is optimal in the budget set  $\{(x, z) \in \mathbb{R}^L \times \mathbb{R} : p^t x + z \leq M^t\}$ , where  $M^t = z^t + p^t x^t$ . Afriat's Theorem also guarantees that this utility function has various nice properties and, in particular,  $\tilde{U}$  can be chosen to be strictly increasing. Let  $\bar{M} = \sum_{t=1}^T M^t / T$  and define the augmented utility function  $U : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  by  $U(x, -e) := \tilde{U}(x, \bar{M} - e)$ . Then since  $\tilde{U}$  is strictly increasing,  $x^t$  solves  $\max_{x \in \mathbb{R}_+^L} U(x, \delta^t - p^t x)$  where  $\delta^t = M^t - \bar{M}$ .

There are two claims in (2). We first consider the case where  $\mathcal{D}$  obeys GAPP. Note that whenever  $p^{t'} x^{t'} - p^t x^t < 0$ , then for any  $\{q^t\}_{t=1}^T$  and  $\{\delta^t\}_{t=1}^T$  such that (7) holds, we obtain  $p^{t'} x^{t'} - p^t x^t < \delta^{t'} - \epsilon^{t'} x^{t'} - \delta^t + \epsilon^t x^t$ . This inequality can be re-written as

$$-q^t x^{t'} + \delta^t < -q^{t'} x^{t'} + \delta^{t'}. \quad (16)$$

Choose a number  $M > \max_t (q^t x^t - \delta^t)$  and define the data set  $\tilde{\mathcal{D}} = \{(q^t, 1), (x^t, M + \delta^t - q^t x^t)\}_{t=1}^T$ . Since (16) holds whenever  $p^{t'} x^{t'} - p^t x^t < 0$ ,

$$(q^t, 1)(x^t, M + \delta^t - q^t x^t) > (q^{t'}, 1)(x^{t'}, M + \delta^{t'} - q^{t'} x^{t'}) \text{ only if } p^{t'} x^{t'} > p^t x^t.$$

This guarantees that  $\tilde{\mathcal{D}}$  obeys GARP since  $\mathcal{D}$  obeys GAPP. By Afriat's Theorem, there is a strictly increasing utility function  $\tilde{U} : \mathbb{R}^L \rightarrow \mathbb{R}$  such that  $(x^t, M + \delta^t - q^t x^t)$  is optimal in the budget set  $\{(x, z) \in \mathbb{R}^{L+1} : q^t x + z \leq M + \delta^t\}$ . Define the augmented utility function  $U : \mathbb{R}_+^L \times \mathbb{R}_- \rightarrow \mathbb{R}$  by  $U(x, -e) := \tilde{U}(x, M - e)$ . Since  $\tilde{U}$  is strictly increasing,  $x^t$  solves  $\max_{x \in \mathbb{R}_+^L} U(x, \delta^t - q^t x)$ .

Now consider the case where  $\mathcal{D}$  violates GAPP. Observe that whenever  $p^{t'} x^{t'} - p^t x^t > 0$ , then for any  $\{q^t\}_{t=1}^T$  and  $\{\delta^t\}_{t=1}^T$  such that (7) holds, we obtain  $p^{t'} x^{t'} - p^t x^t > \delta^{t'} - \epsilon^{t'} x^{t'} - \delta^t + \epsilon^t x^t$ . This inequality can be re-written as

$$-q^t x^{t'} + \delta^t > -q^{t'} x^{t'} + \delta^{t'}. \quad (17)$$

By way of contradiction, suppose there is an augmented utility function  $U$  such that  $x^t$  maximizes  $U(x, -q^t x + \delta^t)$  for all  $t$ . For any observation  $t$ , we write  $V^t := \max_{x \in X} U(x, -q^t x + \delta^t)$ . Since (17) holds, we obtain

$$V^t \geq U(x^{t'}, -q^t x^{t'} + \delta^t) > U(x^{t'}, -q^{t'} x^{t'} + \delta^{t'}) = V^{t'}.$$

Thus, we have shown that  $V^t > V^{t'}$  whenever  $p^{t'} x^{t'} - p^t x^t > 0$ . Since  $\mathcal{D}$  violates GAPP there is a finite sequence  $\{(p^{t_1}, x^{t_1}), \dots, (p^{t_N}, x^{t_N})\}$  of distinct elements in  $\mathcal{D}$ , such that  $p^{t_i} x^{t_{i+1}} < p^{t_{i+1}} x^{t_{i+1}}$  for all  $i \in \{1, \dots, N-1\}$  and  $p^{t_N} x^{t_1} < p^{t_1} x^{t_1}$ . By the observation we have just made, we obtain  $V^{t_1} > V^{t_2} > \dots > V^{t_N} > V^{t_1}$ , which is impossible.  $\blacksquare$

**PROOF OF PROPOSITION 3.**

(1) We have already shown the ‘only if’ part of this claim, so we need to show the ‘if’ part holds. From the proof of [Theorem 1](#), we know that for a large  $M$ , it is the case that  $p^t \succeq_p p^{t'}$  if and only if  $(x^t, M - p^t x^t) \succeq_x (x^{t'}, M - p^{t'} x^{t'})$  and hence  $p^t \succeq_p^* p^{t'}$  if and only if  $(x^t, M - p^t x^t) \succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$ . If  $p^t \not\succeq_p^* p^{t'}$ , then  $(x^t, M - p^t x^t) \not\succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$  and hence there is a utility function  $\tilde{U} : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}$  rationalizing the augmented data set  $\tilde{D}$  such that  $\tilde{U}(x^t, M - p^t x^t) < \tilde{U}(x^{t'}, M - p^{t'} x^{t'})$  (see Remark 2 in [Section C](#)). This in turn implies that the augmented utility function  $U$  (as defined by (3)), has the property that  $U(x^t, -p^t x^t) < U(x^{t'}, -p^{t'} x^{t'})$  or, equivalently,  $V(p^t) < V(p^{t'})$ .

(2) Given part (1), we need only show that if  $p^t \succeq_p^* p^{t'}$  but  $p^t \not\succeq_p^* p^{t'}$ , then there is some augmented utility function  $U$  such that  $U(x^t, -p^t x^t) = U(x^{t'}, -p^{t'} x^{t'})$ . To see that this holds, note that if  $p^t \succeq_p^* p^{t'}$  but  $p^t \not\succeq_p^* p^{t'}$ , then  $(x^t, M - p^t x^t) \succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$  but  $(x^t, M - p^t x^t) \not\succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$ . In this case there is a utility function  $\tilde{U} : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}$  rationalizing the augmented data set  $\tilde{D}$  such that  $\tilde{U}(x^t, M - p^t x^t) = \tilde{U}(x^{t'}, M - p^{t'} x^{t'})$ . This in turn implies that the augmented utility function  $U$  (as defined by (3)) satisfies  $U(x^t, -p^t x^t) = U(x^{t'}, -p^{t'} x^{t'})$  and so  $V(p^t) = V(p^{t'})$ . ■

**PROOF OF THEOREM 2.** We prove this result for the more general case where the consumption space is a closed subset  $X$  of  $\mathbb{R}_+^L$ . It is obvious that (3) implies (1). We first show that (1) implies (2), i.e., nonlinear GAPP is necessary for rationalization. Given a price system  $\psi : X \rightarrow \mathbb{R}$ , we define the indirect utility  $V(\psi) = \max_{x \in X} U(x, -\psi(x))$ . If  $\psi^{t'} \succeq_p \psi^t$ , then  $\psi^{t'}(x^t) \leq \psi^t(x^t)$ , and so

$$V(\psi^{t'}) \geq U(x^t, -\psi^{t'}(x^t)) \geq U(x^t, -\psi^t(x^t)) = V(\psi^t).$$

Furthermore,  $U(x^t, -\psi^{t'}(x^t)) > U(x^t, -\psi^t(x^t))$  if  $\psi^{t'} \succ_p \psi^t$ , and in that case  $V(\psi^{t'}) > V(\psi^t)$ . It follows that if  $p^{t'} \succeq_p^* p^t$ , then  $V(\psi^{t'}) \geq V(\psi^t)$ , which mean that it is not possible for  $p^t \succ_p^* p^{t'}$ .

It remains for us to show that (2) implies (3). Choose a number  $M > \max_{t \in T} \psi^t(x^t)$ . We associate to each observation  $t$  the constraint set

$$C^t = \{(x, z) \in X \times \mathbb{R}_+ : \psi^t(x) + z \leq M\}$$

and define the augmented data set  $\tilde{D} = \{(C^t, (x^t, z^t))\}$ , where  $z^t = M - \psi^t(x^t)$ . Our assumptions on  $\psi^t$  guarantee that  $C^t$  is a compact set and that it is downward comprehensive, i.e., if  $(x, z) \in C^t$  then so is  $(x', z') \in X \times \mathbb{R}_+$  such that  $(x', z') \leq (x, z)$ .

Notice that  $(x^t, z^t) \in C^{t'}$  if and only if  $\psi^{t'} \succeq_p \psi^t$ . This is because  $(x^t, z^t) \in C^{t'}$  if and only if

$$\psi^{t'}(x^t) + z^t \leq M = \psi^t(x^t) + z^t.$$

When this occurs, we write  $(x^{t'}, z^{t'}) \succeq_x (x^t, z^t)$ . Since  $\psi^t$  is increasing, if  $(\hat{x}, \hat{z}) > (x, z)$ , then  $\psi(\hat{x}) + \hat{z} > \psi(x) + z$ . This guarantees that  $(x^t, z^t) \in \underline{C}^{t'}$  if and only if  $\psi^{t'} \succ_p \psi^t$ , where  $\underline{C}^{t'}$  consists of those elements in  $C^{t'}$  which are dominated by some element in  $C^{t'}$ , i.e.,  $(x, z) \in \underline{C}^{t'}$  if it is in  $C^{t'}$  and there is  $(\hat{x}, \hat{z}) \in C^{t'}$  such that  $(\hat{x}, \hat{z}) > (x, z)$ ; in this case, we write  $(x^{t'}, z^{t'}) \succ_x (x^t, z^t)$ . Given  $\succeq_x$  and  $\succ_x$ , we can then define GARP in the usual way. It is clear that  $\tilde{\mathcal{D}}$  satisfies GARP if and only  $\mathcal{D}$  satisfies nonlinear GAPP.

Since  $\mathcal{D}$  obeys nonlinear GAPP,  $\tilde{\mathcal{D}}$  obeys GARP and by Theorem 2 in Nishimura, Ok, and Quah (2017), there is a continuous and strictly increasing function  $\tilde{U} : X \times \mathbb{R}_+ \rightarrow \mathbb{R}$  for which  $(x^t, z^t)$  maximizes  $\tilde{U}$  in  $C^t$ , for all  $t \in T$ . Furthermore,  $\tilde{U}$  can be chosen to have the following property: if  $(x, z) \in C^t$  and  $\tilde{U}(x, z) = \tilde{U}(x^t, z^t)$  then, for some  $s \in T$ ,  $(x, z) = (x^s, z^s)$  and  $(x^s, z^s) \succeq_x^* (x^t, z^t)$ . In other words, any bundle  $(x, z)$  in  $C^t$  has strictly lower utility than  $(x^t, z^t)$ , unless it is revealed preferred to  $(x^t, z^t)$  (in which case the two bundles will have the same utility).

Note that we can always choose  $\tilde{U}$  to be bounded above on  $X \times \mathbb{R}_+$ . Indeed, if  $\tilde{U}$  is not bounded, we can define another utility function  $\hat{U}$  on  $X \times \mathbb{R}_+$  by

$$\hat{U}(x, z) = \min\{\tilde{U}(x, z), K + h(x, z)\},$$

where  $K > \max_{t \in T} \{\tilde{U}(x^t, z^t)\}$  and  $h$  is a positive, continuous, strictly increasing, and bounded real-valued function on  $X \times \mathbb{R}_+$ . Then  $\hat{U}$  is continuous, strictly increasing, and bounded above. Furthermore, for all  $(x, z) \in C^t$ ,  $\tilde{U}(x, z) < K$  and so  $\hat{U}(x, z) = \tilde{U}(x, z)$ . Therefore,  $(x^t, z^t)$  maximizes  $\hat{U}(x, z)$  in  $C^t$  and  $\hat{U}(x^t, z^t)$  will be strictly greater  $\hat{U}(x, z)$  for any  $(x, z) \in C^t$  that is not revealed preferred to  $(x^t, z^t)$ .

We define the expenditure-augmented utility function

$$U(x, -e) = \tilde{U}(x, \max\{M - e, 0\}) - B \min\{e - M, 0\},$$

where  $B > 0$ . This function is continuous, strictly increasing, and bounded above. We claim that for  $B$  sufficiently large,  $x^t \in \arg \max_{x \in X} U(x, -\psi^t(x))$ . Clearly, if  $x$  satisfies  $M - \psi^t(x) \geq 0$ , in other words  $(x, M - \psi^t(x)) \in C^t$ , then

$$U(x, -\psi(x)) = \tilde{U}(x, M - \psi(x)) \leq \tilde{U}(x^t, M - \psi(x^t)) = U(x^t, -\psi^t(x^t))$$

for any  $B > 0$ . So if there is no  $B$  obeying the specified condition, there must be a sequence  $B_n \rightarrow \infty$ , and  $x_n$  with  $M - \psi^t(x) < 0$  such that, for all  $n$ ,

$$U(x_n, -\psi^t(x_n)) = \tilde{U}(x_n, 0) - B_n(\psi^t(x_n) - M) > U(x^t, -\psi^t(x^t)) = \tilde{U}(x^t, z^t). \quad (18)$$

Since  $\tilde{U}$  is bounded above and  $B_n \rightarrow \infty$ , (18) implies that  $\psi^t(x_n) \rightarrow M$ . By our assumption on  $\psi^t$ ,  $x_n$  is contained in a bounded set; by taking subsequences if necessary we can assume that  $x_n$  has a limit  $\underline{x}$ , with  $\psi^t(\underline{x}) = M$ . Then, it follows from (18) again that  $\tilde{U}(\underline{x}, 0) \geq \tilde{U}(x^t, z^t)$ . But this is impossible because since  $(\underline{x}, 0) \neq (x^s, z^s)$

for any  $s \in T$  and thus  $\tilde{U}(x, 0) < \tilde{U}(x^t, z^t)$ . We conclude that for  $B$  sufficiently large  $x^t \in \arg \max_{x \in X} U(x, -\psi^t(x))$ . Since the number of observations is finite, we could find a  $B$  for which this holds for all  $t \in T$  (in other words,  $U$  rationalizes  $\mathcal{D}$ ).

Lastly, we need to show that  $\max_{x \in X} U(x, -\psi(x))$  has a solution for any price system  $\psi$ . Since  $U$  is bounded above,  $u^* = \sup_{x \in X} U(x, -\psi(x))$  exists and there is a sequence  $x_n$  such that  $U(x_n, -\psi(x_n)) \rightarrow u^*$ . If  $x_n$  is unbounded,  $\psi(x_n) \rightarrow \infty$ , which (given the form of  $U$ ) means that  $U(x_n, -\psi(x_n)) \rightarrow -\infty$ . This is impossible and so  $x_n$  is bounded and will have a subsequence converging to some  $x^* \in X$ . By the continuity of  $U$  and  $\psi$ , we obtain  $U(x^*, -\psi(x^*)) = u^*$ . ■

#### APPENDIX B. OMITTED MATERIAL FROM SECTION 4

In this section, we formally develop our bootstrap procedure from Section 4.2. We begin by describing Weyl-Minkowski duality<sup>25</sup> which is used for the equivalent (dual) restatement (15) of our test (14). As we mentioned earlier, we will also appeal to this duality in the proof of the asymptotic validity of our testing procedure.

**Theorem 5.** (*Weyl-Minkowski Theorem for Cones*) *A subset  $\mathcal{C}$  of  $\mathbb{R}^I$  is a finitely generated cone*

$$\mathcal{C} = \{v_1 a_1 + \dots + v_{|A|} a_{|A|} : v_h \geq 0\} \text{ for some } A = [a_1, \dots, a_H] \in \mathbb{R}^{I \times |A|} \quad (19)$$

*if, and only if, it is a finite intersection of closed half spaces*

$$\mathcal{C} = \{t \in \mathbb{R}^I \mid Bt \leq 0\} \text{ for some } B \in \mathbb{R}^{m \times I}. \quad (20)$$

The expressions in (19) and (20) are called a  $\mathcal{V}$ -representation (as in “vertices”) and a  $\mathcal{H}$ -representation (as in “half spaces”) of  $\mathcal{C}$ , respectively. In what follows, we use an  $\mathcal{H}$ -representation of cone( $A$ ) corresponding to a  $m \times I$  matrix  $B$  as implied by Theorem 5.

We are now in a position to show that the bootstrap procedure defined in Section 4.2 is asymptotically valid. Note first that  $\Theta = [\underline{\theta}, \bar{\theta}]$ , where

$$\bar{\theta} = \max_{v \in \Delta^{|A|-1}} \rho v = \max_{1 \leq j \leq |A|} \rho_j \quad (21)$$

$$\underline{\theta} = \min_{v \in \Delta^{|A|-1}} \rho v = \min_{1 \leq j \leq |A|} \rho_j, \quad (22)$$

where  $\rho_j$  denotes the  $j$ th component of  $\rho$ . We normalize  $(\rho, \theta)$  such that  $\Theta = [\underline{\theta}, \underline{\theta} + 1]$ . Next, define

$$\mathcal{H} := \{1, 2, \dots, |A|\} \quad (23)$$

$$\bar{\mathcal{H}} := \{j \in \mathcal{H} \mid \rho_j = \bar{\theta}\} \quad (24)$$

$$\underline{\mathcal{H}} := \{j \in \mathcal{H} \mid \rho_j = \underline{\theta}\} \quad (25)$$

<sup>25</sup>See, for example, Theorem 1.3 in Ziegler (1995).

$$\mathcal{H}_0 := \mathcal{H} \setminus (\overline{\mathcal{H}} \cup \underline{\mathcal{H}}). \quad (26)$$

Recall that  $\tau_N$  is a tuning parameter chosen such that  $\tau_N \downarrow 0$  and  $\sqrt{N}\tau_N \uparrow \infty$ . For  $\theta \in \Theta_I$ , we now formally define the  $\tau_N$ -tightened version of  $\mathcal{S}$  as

$$\mathcal{S}_{\tau_N}(\theta) := \{Av \mid \rho v = \theta, v \in \mathcal{V}_{\tau_N}(\theta)\},$$

where

$$\mathcal{V}_{\tau_N}(\theta) := \left\{ v \in \Delta^{|A|-1} \left| \begin{array}{l} v_j \geq \frac{(\bar{\theta} - \theta)\tau_N}{|\underline{\mathcal{H}} \cup \mathcal{H}_0|}, j \in \underline{\mathcal{H}}, v_{j'} \geq \frac{(\theta - \underline{\theta})\tau_N}{|\overline{\mathcal{H}} \cup \mathcal{H}_0|}, j' \in \overline{\mathcal{H}}, \\ v_{j''} \geq \left[ 1 - \frac{(\bar{\theta} - \theta)|\underline{\mathcal{H}}|}{|\underline{\mathcal{H}} \cup \mathcal{H}_0|} - \frac{(\theta - \underline{\theta})|\overline{\mathcal{H}}|}{|\overline{\mathcal{H}} \cup \mathcal{H}_0|} \right] \frac{\tau_N}{|\mathcal{H}_0|}, j'' \in \mathcal{H}_0 \end{array} \right. \right\}.$$

In applications where  $\rho$  is binary, the above notation simplifies. Specifically, in our empirical application on deriving the welfare bounds,  $\rho = \mathbb{1}_{t \geq p^t}$  and  $\theta = \mathcal{N}_{t \geq p^t}$ . Here,  $\bar{\theta} = 1, \underline{\theta} = 0$ , and  $\bar{\theta} - \underline{\theta} = 1$  holds without any normalization. Also,  $\overline{\mathcal{H}}$  ( $\underline{\mathcal{H}}$ ) is just the set of indices for the types that (do not) prefer price  $p^t$  compared to  $p^{t'}$ , while  $\mathcal{H}_0$  is empty. We then have:

$$\mathcal{S}_{\tau_N}(\mathcal{N}_{t \geq p^t}) = \left\{ Av \mid \mathbb{1}'_{t \geq p^t} v = \mathcal{N}_{t \geq p^t}, v \in \mathcal{V}_{\tau_N}(\mathcal{N}_{t \geq p^t}) \right\},$$

where

$$\mathcal{V}_{\tau_N}(\mathcal{N}_{t \geq p^t}) = \left\{ v \in \Delta^{|A|-1} \left| v_j \geq \frac{(1 - \mathcal{N}_{t \geq p^t})\tau_N}{|\underline{\mathcal{H}}|}, j \in \underline{\mathcal{H}}, v_{j'} \geq \frac{\mathcal{N}_{t \geq p^t}\tau_N}{|\overline{\mathcal{H}}|}, j' \in \overline{\mathcal{H}} \right. \right\}.$$

We now state the mild data assumptions.

**Assumption 2.** For all  $t = 1, \dots, T$ ,  $\frac{N_t}{N} \rightarrow \kappa_t$  as  $N \rightarrow \infty$ , where  $\kappa_t > 0, 1 \leq t \leq T$ .

**Assumption 3.** The econometrician observes  $T$  independent cross-sections of i.i.d. samples  $\left\{ x_{n(t)}^t \right\}_{n(t)=1}^{N_t}, t = 1, \dots, T$  of consumers' choices corresponding to the known price vectors  $\{p_t\}_{t=1}^T$ .

Next, let  $\mathbf{d}_{n(t)}^{i,t} := \mathbf{1}\{x_{n(t)}^t \in B^{i,t}\}$ ,  $\mathbf{d}_{n(t)}^t = [\mathbf{d}_{n(t)}^{1,t}, \dots, \mathbf{d}_{n(t)}^{I,t}]$ , and  $\mathbf{d}_n^t = [\mathbf{d}_n^{1,t}, \dots, \mathbf{d}_n^{I,t}]$ . Let  $\mathbf{d}_t$  denote the choice vector of a consumer facing price  $p^t$  (we can, for example, let  $\mathbf{d}_t = \mathbf{d}_1^t$ ). Define  $\mathbf{d} = [\mathbf{d}'_1, \dots, \mathbf{d}'_T]'$ : note,  $E[\mathbf{d}] = \pi$  holds by definition. Among the rows of  $B$  some of them correspond to constraints that hold trivially by definition, whereas some are for non-trivial constraints. Let  $\mathcal{K}^R$  be the index set for the latter. Finally, let

$$\begin{aligned} g &= B\mathbf{d} \\ &= [g_1, \dots, g_m]'. \end{aligned}$$

With these definitions, consider the following requirement:



**Condition 1.** For each  $k \in \mathcal{K}^R$ ,  $\text{var}(g_k) > 0$  and  $E[|g_k / \sqrt{\text{var}(g_k)}|^{2+c_1}] < c_2$  hold, where  $c_1$  and  $c_2$  are positive constants.

This guarantees the Lyapunov condition for the triangular array CLT used in establishing asymptotic uniform validity. This type of condition has been used widely in the literature of moment inequalities; see [Andrews and Soares \(2010\)](#).

**PROOF OF THEOREM 4.**

Define

$$\mathcal{C} = \text{cone}(A)$$

and

$$\mathcal{T}(\theta) = \{\pi = Av : \rho'v = \theta, v \in \mathbb{R}^{|A|}\},$$

an affine subspace of  $\mathbb{R}^I$ . It is convenient to rewrite  $\mathcal{T}(\theta)$  as  $\mathcal{T}(\theta) = \{t \in \mathbb{R}^I : \tilde{B}t = d(\theta)\}$  where  $\tilde{B} \in \tilde{m} \times \mathbb{R}^I$ ,  $d(\cdot) \in \tilde{m} \times 1$ , and  $\tilde{m}$  all depend on  $(\rho, A)$ . We let  $\tilde{b}_j$  denote the  $j$ -th row of  $\tilde{B}$ . Then

$$\mathcal{S}(\theta) = \mathcal{C} \cap \Delta^{|A|-1} \cap \mathcal{T}(\theta).$$

By [Theorem 5](#),  $\mathcal{C} = \{\pi : B\pi \leq 0\}$ , therefore

$$\mathcal{S}(\theta) = \{t \in \mathbb{R}^{|A|} : Bt \leq 0, \tilde{B}t = d(\theta), \mathbf{1}'_H t = 1\}. \quad (27)$$

Let

$$\psi(\theta) = [\psi_1(\theta), \dots, \psi_H(\theta)]' \quad \theta \in \Theta$$

with

$$\psi_j(\theta) = \begin{cases} \frac{(\bar{\theta} - \theta)}{|\mathcal{H} \cup \mathcal{H}_0|} & \text{if } j \in \mathcal{H} \\ \frac{(\theta - \bar{\theta})}{|\mathcal{H} \cup \mathcal{H}_0|} & \text{if } j \in \bar{\mathcal{H}} \\ \left[ 1 - \frac{(\theta - \theta)|\mathcal{H}|}{|\mathcal{H} \cup \mathcal{H}_0|} - \frac{(\theta - \bar{\theta})|\bar{\mathcal{H}}|}{|\mathcal{H} \cup \mathcal{H}_0|} \right] \frac{1}{|\mathcal{H}_0|} & \text{if } j \in \mathcal{H}_0 \end{cases},$$

where terms are defined in [\(23\)](#)-[\(26\)](#). Then

$$\mathcal{S}_{\tau_N}(\theta) = \{\pi = Av : v \geq \tau_N \psi(\theta), v \in \Delta^{|A|-1}, \rho'v = \theta\}.$$

Finally, let

$$\mathcal{C}_{\tau_N} = \{\pi = Av : v \geq \tau_N \psi(\theta)\}.$$

Then

$$\mathcal{S}_{\tau_N}(\theta) = \mathcal{C}_{\tau_N} \cap \Delta^{|A|-1} \cap \mathcal{T}(\theta).$$

Proceeding as in the proof of [Lemma 4.1](#) in [KS](#), we can express the set  $\mathcal{C}_{\tau_N}$  as

$$\mathcal{C}_{\tau_N} = \{t : Bt \leq -\tau_N \phi(\theta)\}$$

where

$$\phi(\theta) = -BA\psi(\theta).$$

As in Lemma 4.1 in [KS](#), let the first  $\bar{m}$  rows of  $B$  represent inequality constraints and the rest equalities, and also let  $\Phi_{kh}$  the  $(k, h)$ -element of the matrix  $-BA$ . We have

$$\phi_k = \sum_{h=1}^{|A|} \Phi_{kh} \psi_h(\theta)$$

where, for each  $k \leq \bar{m}$ ,  $\{\Phi_{kh}\}_{h=1}^{|A|}$  are all nonnegative, with at least some of them being strictly positive, and  $\Phi_{kh} = 0$  for all  $h$  if  $\bar{m} < k \leq m$ . Since  $\psi_h(\theta) > 0, 1 \leq h \leq |A|$  for every  $\theta \in \Theta$  by definition, we have  $\phi_j(\theta) \geq C, 1 \leq j \leq \bar{m}$  for some positive constant  $C$ , and  $\phi_j(\theta) = 0, \bar{m} < j \leq m$  for every  $\theta \in \Theta$ . Putting these together, we have

$$\mathcal{S}_{\tau_N}(\theta) = \{t \in \mathbb{R}^{|A|} : Bt \leq -\tau_N \phi(\theta), \tilde{B}t = d(\theta), \mathbf{1}'_H t = 1\}$$

where  $\mathbf{1}_H$  denotes the  $|A|$ -vector of ones. Define the  $\mathbb{R}^I$ -valued random vector

$$\pi_{\tau_N}^* := \frac{1}{\sqrt{N}} \zeta + \hat{\eta}_{\tau_N}, \quad \zeta \sim N(0, \hat{S})$$

where  $\hat{S}$  is a consistent estimator for the asymptotic covariance matrix of  $\sqrt{N}(\hat{\pi} - \pi)$ . Then (conditional on the data) the distribution of

$$\delta^*(\theta) := N \min_{\eta \in \mathcal{S}_{\tau_N}(\theta)} [\pi_{\tau_N}^* - \eta]' \Omega [\pi_{\tau_N}^* - \eta]$$

corresponds to that of the bootstrap test statistics. Let

$$B_* := \begin{bmatrix} B \\ \tilde{B} \\ \mathbf{1}'_H \end{bmatrix}$$

Define  $\ell = \text{rank}(B_*)$  for the augmented matrix  $B_*$  instead of  $B$  in [KS](#), and let the  $\ell \times m$ -matrix  $K$  be such that  $KB_*$  is a matrix whose rows consist of a basis of the row space of  $(B_*)$ . Also let  $M$  be an  $(I - \ell) \times I$  matrix whose rows form an orthonormal basis of  $\ker B_* = \ker(KB_*)$ , and define  $P = \begin{pmatrix} KB_* \\ M \end{pmatrix}$ . Finally, let  $\hat{g} = B_* \hat{\pi}$ .

Define

$$\begin{aligned} T(x, y) &:= \begin{pmatrix} x \\ y \end{pmatrix}' P^{-1'} \Omega P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathbb{R}^\ell, y \in \mathbb{R}^{I-\ell} \\ t(x) &:= \min_{y \in \mathbb{R}^{I-\ell}} T(x, y) \\ s(g) &:= \min_{\gamma = [\gamma^{\leq'}, \gamma^{\leq'}]', \gamma^{\leq} \leq 0, \gamma' \in \text{col}(B)} t(K[g - \gamma]) \end{aligned}$$

with

$$\gamma^{\leq} = \begin{bmatrix} \mathbf{0}_{m-\bar{m}} \\ d(\theta) \\ 1 \end{bmatrix}$$

where  $\mathbf{0}_{m-\bar{m}}$  denotes the  $(m - \bar{m})$ -vector of zeros. It is easy to see that  $t : \mathbb{R}^\ell \rightarrow \mathbb{R}_+$  is a positive definite quadratic form. By (27), we can write

$$\delta_N(\theta) = Ns(\hat{g}) = s(\sqrt{N}\hat{g}).$$

Likewise, for the bootstrapped version of  $\delta$  we have

$$\begin{aligned} \delta^*(\theta) &= N \min_{\eta \in \mathcal{S}_{\tau_N}(\theta)} [\pi_{\tau_N}^* - \eta]' \Omega [\pi_{\tau_N}^* - \eta] \\ &= s(\varphi_N(\hat{\xi}) + \zeta), \end{aligned}$$

where  $\hat{\xi} = B_* \hat{\pi} / \tau_N$ . Note the function  $\varphi_N(\xi) = [\varphi_N^1(\xi), \dots, \varphi_N^m(\xi)]$  for  $\xi = (\xi_1, \dots, \xi_m)' \in \text{col}(B_*)$ . Moreover, its  $k$ -th element  $\varphi_N^k$  for  $k \leq \bar{m}$  satisfies

$$\varphi_N^k(\xi) = 0$$

if  $|\xi^k| \leq \delta$  and  $\xi^j \leq \delta, 1 \leq j \leq m, \delta > 0$ , for large enough  $N$  and  $\varphi_N^k(\xi) = 0$  for  $k > \bar{m}$ . This follows (we use some notation in the proof of Theorem 4.2 in KS, which the reader is referred to) by first noting that it suffices to show that for small enough  $\delta > 0$ , every element  $x^*$  that fulfills equation (9.2) in KS with its RHS intersected with  $\cap_{j=1}^{\bar{m}} \tilde{S}_j(\delta), \tilde{S}_j(\delta) = \{x : |\tilde{b}'_j x - d_j(\theta)| \leq \tau\delta\}$  satisfies

$$x^* | \mathcal{S}(\theta) \in \cap_{j=1}^q H_j^\tau \cap L \cap \mathcal{T}(\theta).$$

If not, then there exists  $(\tilde{a}, \tilde{x}) \in F \cap \mathcal{T}(\theta) \times \cap_{j=1}^q H_j \cap L \cap \mathcal{T}(\theta)$  such that

$$(\tilde{a} - \tilde{x})'(\tilde{x} | \mathcal{S}_\tau(\theta) - \tilde{x}) = 0,$$

where  $\tilde{x} | \mathcal{S}_\tau(\theta)$  denotes the orthogonal projection of  $\tilde{x}$  on  $\mathcal{S}_\tau(\theta)$ . This, in turn, implies that there exists a triplet  $(a_0, a_1, a_2) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$  such that  $(a_1 - a_0)'(a_2 - a_0) < 0$ . But as shown in the proof of Theorem 4.2 in KS, this cannot happen. The conclusion then follows by Theorem 1 of Andrews and Soares (2010).  $\blacksquare$

## APPENDIX C. NOT FOR PUBLICATION ONLINE APPENDIX: AFRIAT'S THEOREM

**Afriat's Theorem (Afriat (1967)).** *Given a data set  $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ , the following are equivalent:*

- (1)  $\mathcal{D}$  can be rationalized by a locally nonsatiated utility function.
- (2)  $\mathcal{D}$  satisfies GARP.
- (3)  $\mathcal{D}$  can be rationalized by a strictly increasing, continuous, and concave utility function.

REMARK 1. The discussion preceding the theorem essentially shows that (1) implies (2). The standard argument that shows (2) implies (3) (for instance in [Fostel, Scarf, and Todd \(2004\)](#)) works by showing that a consequence of GARP is that there exist numbers  $\phi^t$  and  $\lambda^t > 0$  (for all  $t \in T$ ) that solve the inequalities

$$\phi^{t'} \leq \phi^t + \lambda^t p^t(x^{t'} - x^t) \text{ for all } t' \neq t. \quad (28)$$

It is then straightforward to show that

$$\tilde{U}(x) = \min_{t \in T} \{\phi^t + \lambda^t p^t(x - x^t)\} \quad (29)$$

rationalizes  $\mathcal{D}$ , with the utility of the observed consumption bundles satisfying  $\tilde{U}(x^t) = \phi^t$ . The function  $\tilde{U}$  is the lower envelope of a finite number of strictly increasing affine functions, and so it is strictly increasing, continuous, and concave. A remarkable feature of this theorem is that while GARP follows simply from local nonsatiation of the utility function, it is sufficient to guarantee that  $\mathcal{D}$  is rationalized by a utility function with significantly stronger properties.

REMARK 2. To be precise, GARP guarantees that there is preference  $\succsim$  (in other words, a complete, reflexive, and transitive binary relation) on  $\mathcal{X}$  that extends the (potentially incomplete) revealed preference relations  $\succeq_x^*$  and  $\succ_x^*$  in the following sense: if  $x^{t'} \succeq_x^* x^t$ , then  $x^{t'} \succsim x^t$  and if  $x^{t'} \succ_x^* x^t$  then  $x^{t'} \succ x^t$ . One could then proceed to show that, for *any* such preference  $\succsim$ , there is in turn a utility function  $\tilde{U}$  that rationalizes  $\mathcal{D}$  and extends  $\succsim$  (from  $\mathcal{X}$  to  $\mathbb{R}_+^L$ ) in the sense that  $\tilde{U}(x^{t'}) \geq (>) \tilde{U}(x^t)$  if  $x^{t'} \succsim (>) x^t$  (see [Quah \(2014\)](#)). This has implications on the inferences one could draw from the data. If  $x^{t'} \not\succeq_x^* x^t$  (or if  $x^{t'} \succeq_x^* x^t$  but  $x^{t'} \not\succeq_x^* x^t$ ) then it is always possible to find a preference extending the revealed preference relations such that  $x^t \succ x^{t'}$  (or  $x^t \sim x^{t'}$  respectively).<sup>26</sup> Therefore,  $x^{t'} \succeq_x^* (\succ_x^*) x^t$  if and only if every locally nonsatiated utility function rationalizing  $\mathcal{D}$  has the property that  $\tilde{U}(x^{t'}) \geq (>) \tilde{U}(x^t)$ . We will similarly argue that the revealed price preference relation contains the most detailed information for welfare comparisons in our model.

<sup>26</sup>We use  $x^{t'} \sim x^t$  to mean that  $x^{t'} \succsim x^t$  and  $x^t \succsim x^{t'}$ .

REMARK 3. A feature of [Afriat's Theorem](#) that is less often remarked upon is that in fact  $\tilde{U}$ , as given by (29), is well-defined, strictly increasing, continuous, and concave on the domain  $\mathbb{R}^L$ , rather than just the positive orthant  $\mathbb{R}_+^L$ . Furthermore,

$$x^t \in \operatorname{argmax}_{\{x \in \mathbb{R}^L: p^t x \leq p^t x^t\}} \tilde{U}(x) \quad \text{for all } t \in T. \quad (30)$$

In other words,  $x^t$  is optimal even if  $\tilde{U}$  is extended beyond the positive orthant and  $x$  can be chosen from the larger domain. (Compare (30) with (1).) We utilize this feature when we apply [Afriat's Theorem](#) in our proofs.