The Alternative-Dependent Focal Luce Model

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Abstract

We introduce the alternative-dependent focal Luce model (ADFLM), which is a random choice model that generalizes the well-known Luce (1959) model. In the ADFLM, focal alternatives are chosen more frequently relative to their utilities. We show that we can identify utilities, focal sets and the magnitude of focal biases from choice data. We also axiomatically characterize the ADFLM by weakening IIA. Our model can explain the well-known behavioral phenomena, the attraction and compromise effects.

Keywords: Random Choice Rule, Alternative-Dependent Focal Luce Model, IIA, Focal Set.

1 Introduction

Luce's (1959) model (or multinomial logit model) is the most famous stochastic choice model, and it is characterized by the independence from irrelevant alternatives (IIA) axiom. The IIA states that the relative probability of choosing one alternative over another from a menu is not affected by the presence of other alternatives. We propose and characterize a generalization of the Luce's model in which focality of alternatives affect choice frequencies.

It is well-known that the IIA may fail in reality when an alternative has different focality across different menus. For example, the probability of a dish being chosen can be very different when you display it on the first page or on the last page of the menu. The dishes on the first page gets more attention than the dishes on the last page since consumers cannot give every dish the same attention when the menu is large.

If a seller has an experience of selling on eBay, he will find that the views of his listing growing fast in the first day, but it decays quickly after the first day. In this period, his listing usually sells fast. This is because his listing belong to "new listing" in the first day, which is usually displayed on the first page of these similar items. But it will no longer belongs to "new listing" after one day, and it is not displayed on the first page anymore. In this period, his listing usually takes a very long time to sell. This phenomena indicates that consumer's attention is limited, the choice is affected by the attention.The limited consideration effect has been studied by Manzini and Mariotti (2007), Masatlioglu, Nakajima and Ozbay (2012), Cherepanov, Feddersen and Sandroni (2013), and Lleras et al. (2017). Kovach and Tserenjigmid (2021) introduced the Focal Luce Model to capture the impact of the focus of consumers. They studied a menu-dependent focal bias for focal alternatives, where focal alternatives receive the same bias in the same menu. Compared to their model, we study an alternative-dependent bias for alternatives, that is the focal biases are the same for the same alternatives in every menu. We call it the Alternative-dependent Focal Luce Model (ADFLM). Formally, a random choice rule p is an ADFLM if there is a utility function u, a focus function F, and a focus bias function δ such that for any $A \in \mathscr{A}$ and $a \in A$,

$$p(a, A) = \frac{u(a)(1 + \delta(a)\mathbb{1}\{a \in F(A)\})}{\sum_{b \in A} u(b)(1 + \delta(b)\mathbb{1}\{b \in F(A)\})}. \tag{1}$$

The ADFLM is not a random utility model, but it can be thought as a random, menu-dependent utility model. Under this view, the random utility of an alternative x in menu A is

$$v(x, A) = \bar{u}(x) + \bar{\delta}(x)\mathbb{1}\{x \in F(A)\} + \epsilon_x. \tag{2}$$

In this equation, $\bar{u}(x)$ is the fixed utility of x and ϵ_x is the random part of utility. The additional utility for a focal alternative is $\bar{\delta}(x)$, which is alternative-dependent. The consumer will choose the alternative with the highest utility v given ϵ_x for all $x \in A$. When ϵ_x follows the standard extreme value type I distribution, as in the multinomial logit, Equation 2 can derive the random choice probability of ADFLM in Equation 1.

The rest of this paper is organized as follows. In Section 2, we define the model, and show that the ADFLM can explain the attraction and compromise effects. In Section 3, we infer the focal set from choice frequencies and axiomatically characterize our model. In Section 4, we compare the ADFLM with Luce model and FLM with some data. In Section 5, we discuss the related literature.

2 Model

2.1 Model Setup

We first introduce the random choice rule and Luce model before introducing our model. Here, we denote the set X as a finite set of all alternatives and $\mathscr{A}$ as the collection of all nonempty sets subsets of X.

Definition 1. A function p:$X \times \mathscr{A} \to [0, 1]$ is called a *random choice rule*, if for any $A \in \mathscr{A}$, $\sum_{x \in A} p(x, A) = 1$, and $p(y, A) = 0$ when $y \notin A$.

In this paper, we only focus on positive random choice rules. i.e., we assume $p(x, A) > 0$ for any $x \in A$. The Luce model is the most well-known random choice rule. It is defined as follows.

Definition 2. A random choice rule p is a Luce Model if there is a utility function v: $X \to \mathbb{R}_{++}$ such that for any $A \in \mathscr{A}$ and $x \in A$,

$$p(x, A) = \frac{v(x)}{\sum_{y \in A} v(y)}.$$

For notational simplification, we define the probability ratio of a and b as $r(a,b) = \frac{p(a,\{a,b\})}{p(b,\{a,b\})}$ for a binary menu with alternatives a and b.We also denote the probability ratio of a and b in a menu A as $r_A(a,b) = \frac{p(a,A)}{p(b,A)}$. Note that the Luce model satisfies the *independence of irrelevant alternatives (IIA)* axiom, which is defined as follows.

Definition 3. A random choice rule p satisfies Luce's independence of irrelevant alternatives (IIA) axiom if for any $A, B \in \mathscr{A}$ and all $a, b \in A \cap B$,

$$r_A(a,b) = r_B(a,b).$$

IIA may fail in reality, for example, the ratio could be very different when both alternatives on the first page of a menu from the ratio when one alternative is on the first page and the second is on the last page. This is because the alternative on the first page draws more attention, that is, the alternative is salient or focal. Now we introduce an alternative-dependent focal Luce model to capture the saliency of alternatives.

Before we introduce our model, we first introduce a focus function that specifies the focal alternatives for each menu. The focus function is a mapping $F\colon \mathscr{A} \to 2^X$ if $F(A) \subseteq A$ for any $A \in \mathscr{A}$. We denote the set of focal alternatives as the focal set $F(A)$, where $F(A) \subseteq A$. Then, the alternative-dependent focal Luce model (ADFLM) is defined as follows:

Definition 4. A random choice rule p is an ADFLM if there is a utility function $u\colon X \to \mathbb{R}_{++}$, a focus function $F : \mathscr{A} \to 2^X \setminus \{\phi\}$, and a focus bias function δ: $X \to \mathbb{R}_{++}$ such that for any $A \in \mathscr{A}$ and $a \in A$,

$$p(a,A) = \frac{u(a)(1+\delta(a)\mathbb{1}\{a \in F(A)\})}{\sum_{b \in A} u(b)(1+\delta(b)\mathbb{1}\{b \in F(A)\})}. \tag{3}$$

For a binary menu $\{a,b\}$, it is easy for both alternatives to draw the same attention since there are only two alternatives. Thus, we assume that a, b have same attention for any $a, b \in \{a,b\}$. Neither of them draws any additional attention, that is, both a and b are non-focal in a binary menu ($F(\{a,b\}) = \phi$). Then, $p(a,\{a,b\}) = \frac{u(a)}{u(a)+u(b)}$.

Unlike the FLM of Kovach and Tserenjigmid (2021), which has a menu-dependent focality. The ADFLM is alternative-dependent. Here we define the FLM as follows.

Definition 5. A random choice rule p is a Focal Luce Model (FLM) if there is a utility function u: $X \to \mathbb{R}_{++}$, a focus function $F : \mathscr{A} \to 2^X \setminus \{\phi\}$, and a focus bias function δ: $\mathscr{A} \to \mathbb{R}_{++}$ such that for any $A \in \mathscr{A}$ and $a \in A$,

$$p(a, A) = \frac{u(a)(1 + \delta(A)\mathbb{1}\{a \in F(A)\})}{\sum_{b \in A} u(b)(1 + \delta(A)\mathbb{1}\{b \in F(A)\})}.$$

Before we characterize the ADFLM, we would like to define the Luce Product Rule, which will be used to characterize our ADFLM. LPR is a weakening of IIA, which is only satisfied in binary menus.

Definition 6. (LPR). A random choice rule p satisfies Luce's Product Rule (LPR) if for any $a, b, c \in X$, $r(a, c) = r(a, b) \cdot r(b, c)$.

2.2 Discussion

The ADFLM can explain two well-known behavior phenomena, the attraction effect and compromise effect.[1]

Attraction Effect: The attraction effect is the phenomena that people have a choice reversal when an expensive alternative with medium-quality is introduced to the menu. This is a well-known violation of regularity which says the choice probability of an alternative is not affected by adding a new alternative.

Simonson and Tversky (1992) documented this phenomena in their experimental study. In their experiment, they have 3 microwave ovens x y and z. There were $\{x, y\}$ in the original menu, where x is Emerson with 35% off discount, and y is Panasonic I with 35% off discount. Then, they introduced z, which is Panasonic II with only a 10% off discount. They found that y was chosen with a higher frequency when z was introduced. This phenomena violates the regularity, which cannot be explained by any random utility models including the Luce model. Table 1 shows the choice frequency data from this experiment.

The ADFLM can explain the attraction effect. Suppose for the original menu, $F(\{x, y\}) = \phi$, but $F(\{x, y, z\}) = \{y\}$. In the menu $\{x, y, z\}$, Panasonic I is more

[1]The attraction and compromise effects were first introduced by Simonson (1989) and Huber et al. (1982) respectively in their experimental studies. These phenomena were confirmed by many studies later (e.g., Simonson and Tversky 1992, Tversky and Simonson 1993, Ariely and Wallsten 1995, Herne 1998, Doyle et al. 1999, Chernev 2004, and Sharpe et al. 2008).

Alternatives/Menus	$\{x, y\}$	$\{x, y, z\}$
x	0.57	0.27
y	0.43	0.60
z	–	0.13

Table 1: Choice Frequency for Microwave Oven

attractive than Panasonic II since it has a higher discount. The higher discount draws more attention for y even there are more alternatives in the menu. The attraction effect violates the regularity where $p(y, \{x, y, z\}) = 0.60 > p(y, \{x, y\}) = 0.43$. We can obtain choice frequencies in Table 1 with parameters $u(x) = 1$ (normalized), $u(y) = 0.75$, $u(z) = 1.03$, and $\delta(y) = 1.22$ based on the above data.

Compromise effect: The compromise effect says that people prefer to choose middle option when there are extreme alternatives. Simonson and Tversky (1992) also documented this phenomena in their experimental study. In their experiment, they have 3 versions of 35mm Minolta Cameras, x, y and z. At the original menu $\{x, y\}$, there were x with a low quality and low price, and y with a middle quality and middle price. Then, they added another alternative z with high quality and high price. Then, they found that the probability of choosing y increased (relative to x) when z is added. This phenomena violets the IIA, which cannot be explained by the Luce model. Table 2 shows the choice frequency data from this experiment.

Alternatives/Menus	$\{x, y\}$	$\{x, y, z\}$
x	0.50	0.22
y	0.50	0.57
z	–	0.21

Table 2: Choice Frequency for Minolta Camera

The ADFLM can explain the compromise effect. For the original menu, $F(\{x, y\}) = \phi$, but $F(\{x, y, z\}) = \{y\}$. In the menu $\{x, y, z\}$, the middle alternative y is focal, and the extreme alternatives x and z are non-focal. Mathematically, $p(x, \{x, y\}) > p(x, \{x, y, z\})$ and $p(y, \{x, y\}) < p(y, \{x, y, z\})$ since

$$p(x, \{x, y\}) = \frac{u(x)}{u(x) + u(y)} > \frac{u(x)}{u(x) + u(y) + u(z)} = p(x, \{x, y, z\}),$$

and,

$$p(y, \{x, y\}) = \frac{u(y)}{u(x) + u(y)} < \frac{u(y)(1 + \delta(y))}{u(x) + u(y)(1 + \delta(y)) + u(z)} = p(y, \{x, y, z\}).$$

Table 2 also violates the regularity where $p(y, \{x, y, z\}) = 0.57 > p(y, \{x, y\}) = 0.5$. We can obtain choice frequencies in Table 2 with parameters $u(x) = 1$ (normalized), $u(y) = 1$ $u(z) = 0.96$, and $\delta(y) = 1.59$ based on the above data.

3 Axiomatic Characterization of the Model

In this section, we will axiomatically characterize our model. Before doing that, we first infer the focal set by some probability ratio relationship other than the IIA. This is because IIA fails when some alternatives are in the focal sets.

Definition 7. For any set A, (i) a is revealed focal in A if there exists $c \in A$ such that $r_A(c, a) < r(c, a)$. We denote the set of revealed focal alternatives in A by $F^*(A)$. (ii) a is revealed non-focal in A if for all $c' \in A$, if $r_A(c', a) \geq r(c', a)$. The set of revealed non-focal alternatives in A is denoted by $N^*(A)$, i.e. $N^*(A) = A/F^*(A)$.

We first show that our Revealed Focality correctly identifies focality of alternatives in the ADFLM.

Proposition 1. (Revealed Focality) In the ADFLM, $F^* = F$.

Proof. Since we know that $r_A(c, a) = \frac{(1+\delta(c)\mathbb{1}\{c\in F(A)\})u(c)}{(1+\delta(a)\mathbb{1}\{a\in F(A)\})u(a)}$ and $r(c, a) = \frac{u(c)}{u(a)}$. There exists $c \in A$ such that $r_A(c, a) < r(c, a)$ means a is focal in A. This can derived from the observation that

$$r_A(c, a) < r(c, a) \text{ iff } \frac{1 + \delta(c)\mathbb{1}\{c \in F(A)\}}{1 + \delta(a)} < 1,$$

where either $c \in F(A)$ or $c \in F(A)$ and $\delta(c) < \delta(a)$. The latter inequality indicates that $a \in F(A) = F^*(A)$. Similarly, for all $c' \in A$,

$$r_A(c', a) \geq r(c', a) \text{ iff } 1 + \delta(c')\mathbb{1}\{c' \in F(A)\} \geq 1,$$

where the latter inequality indicates $a \notin F(A)$, but $a \in N(A) = A/F(A) = N^*(A)$. □

Revealed Focality can be used to infer the focal of alternatives in a menu, and is essential for characterizing the ADFLM. We now can define our characterizing axioms.

Axiom 1. (Menu-Independent Focality) *For any $A, B \in \mathscr{A}$ and a, b, c such that $a \in F^*(A) \cap F^*(B)$, $b \in N^*(A)$, and $c \in N^*(B)$,*

$$\frac{r_A(a,b)}{r(a,b)} = \frac{r_B(a,c)}{r(a,c)}.$$

This axiom requires that the effects of focality are menu-independent. It states that if an alternative is focal in two different menus, then the probability ratio between this focal alternative and any other non-focal alternatives over the binary probability ratio does not change across menus.

In this model, we want to show how changes in focus affect choice behavior. IIA is violated in this model, but we can have Revealed IIA, which says alternatives satisfy IIA if they are non-focal in the menu, which is a weakening form of IIA.

Axiom 2. (Revealed IIA) *For any $A \in \mathscr{A}$ and any $a, b \in N^*(A)$, $r_A(a,b) = r(a,b)$.*

Revealed IIA suggests that if a and b are both non-focal in A, then they satisfy IIA. Based on the Revealed IIA, Menu-Independent Focality and LPR we can characterize the ADFLM. Then we have Theorem 1.

Theorem 1. *A random choice rule p satisfies Revealed IIA, Menu-Independent Focality, and LPR if and only if it is an ADFLM.*

Theorem 1 fully characterizes our ADFLM. The proof of Theorem 1 in Appendix A.1. We can also see that u, F and δ are uniquely identified. This is shown in Proposition 2.

Proposition 2. (Uniqueness) *If (u, F, δ) is an ADFLM that represents p, then (u', F', δ') is another ADFLM representation of p if and only if (i) $F = F'$, (ii) there is an $\alpha > 0$, such that $u' = \alpha u$, and (iii) $\delta = \delta'$.*

Since the ADFLM coincides with Luce's model on binary menus, the uniqueness of u is straightforward. The uniqueness of the focal set F follows from Proposition 1. For the focal bias δ, $\frac{r_A(a,b)}{r(a,b)} = 1 + \delta(a)$ since for any A, B and $a, b \in A \cap B$ with $a \in F^*(A)$ and $b \in N^*(A)$, $\delta(a)$ is unique when Menu-Independent Focality satisfied in A and B.

4 Comparison to Luce Model and FLM

In this section we show that the ADFLM is neither a special case of the FLM nor nests of it. To see this, consider examples of the FLM and ADFLM in Table 3 and 4.

Table 3: Example for FLM

Alternatives/ Menus	A	B	C
a_1	1/4	3/9	2/4
a_2	1/4	3/9	1/4
a_3	1/4	2/9	1/4
a_4	1/8	1/9	–
a_5	1/8	–	–

Table 4: Example for ADFLM

Alternatives/ Menus	A	B	C
a_1	1/4	3/10	3/7
a_2	1/4	4/10	2/7
a_3	1/4	2/10	2/7
a_4	1/8	1/10	–
a_5	1/8	–	–

In Table 3 and Table 4, alternatives in menu A are non-focal. For simplicity, we normalize the utility of a_1 to 1, that is $u(a_1) = 1$. Here we can derive that $a_1, a_2 \in F(B), F(C)$. This is because $r_A(a_1, a_2) = r_B(a_1, a_2) = r_C(a_1, a_2) = 1$, but $r_A(a_1, a_3) = 1 \neq r_B(a_1, a_3) = 1.5 \neq r_C(a_1, a_3) = 2$. This means $a_1, a_2 \in F(B), F(C)$, and we can further derive that $\delta(B) = 0.5$ and $\delta(C) = 1$ by the random choice rule p in FLM. Similarly, we can derive that $a_1, a_2 \in F(B)$ and $a_1 \in F(C)$ from Table 3. This is because $r_B(a_1, a_3) = r_C(a_1, a_3) = 1.5$, $r_B(a_1, a_4) = r_C(a_1, a_4) = 3$ and $r_B(a_1, a_5) = r_C(a_1, a_5) = 3$, the focality of a_1, a_3, a_4, a_5 are the same in B and C. But $r_B(a_1, a_2) = 3/4 \neq r_C(a_1, a_2) = 1.5$. This means $a_2 \in F(B)$, but $a_2 \in N(C)$. According to Equation 3, we can derive $\delta(a_1) = 0.5$ and $\delta(a_2) = 1$.

Note that the example in Table 4 cannot be rationalized by FLM. This is because $p(a_1, B) > p(a_1, A)$ and $p(a_2, B) > p(a_2, A)$, which indicates that both a_1 and a_2 are focal in B. Since $u(a_1) = u(a_2)$, $p(a_1, B) = p(a_2, B)$ if the data can be rationalized by FLM. However, $p(a_1, B) = 3/10 \neq p(a_2, B) = 4/10$, it cannot be rationalized by FLM.

By comparing Table 3 and 4 of FLM and ADFLM, we can see that IIA satisfies when alternatives are both focal or non-focal in FLM. For ADFLM, IIA satisfies when alternatives have the same focality in two menus.

5 Related Literature

The benchmark economic model of rational behavior for random choice is the random utility model. The random utility model has a very broad literature, such as Falmagne (1978), Barbera and Pattanaik (1986), and McFadden and Richter (1990). The Luce (1959) Model is a special case of the random utility model. Our model, like Kovach and Tserenjigmid's (2021) Focal Luce model, is not a special case of the random utility model, which can explain the attraction and compromise effects that a random utility model cannot explain.

The ADFLM can be treated as a stochastic version of consideration sets model which has been studied by Manzini and Mariotti (2007), Masatlioglu, Nakajima and Ozbay (2012), Cherepanov, Feddersen and Sandroni (2013), and Lleras et al. (2017). In the consideration sets model, consumers only make their choices among alternatives in the consideration set with a deterministic preference relationship. However, in our model, we allow consumer choose alternatives outside the focal set with a random probability rule p. This is more realistic since individuals may have random preferences. We characterize the ADFLM and identify both the focal set F and the magnitude of the bias δ.

Like the FLM introduced by Kovach and Tserenjigmid (2021), we focus on the focal set and the bias on a focal alternative. Their FLM assumes that the focal bias is menu-dependent. However, we allow for an alternative-dependent focal bias. The ADFLM can explain the additional utility brought by both the focal set and the alternative itself, while the FLM can explain the menu-dependent focal bias.

In this paper, we study the alternative-dependent focal Luce model, where the focal bias is alternative-dependent. We first define our ADFLM, and compare our model with Luce model and FLM. Then, we infer the focal set and axiomatically characterized the model. Compared to the FLM which satisfies the IIA when both alternatives are focal or non-focal on two menus, our ADFLM satisfies the IIA when alternatives are the same in different menus.

Appendix.

A.1. Proof of Theorem 1.

We prove theorem 1 by two parts, both the necessity and sufficiency.
Necessity. Given the ADFLM p, we need to prove that p satisfy LPR, Revealed IIA and Menu-Independent Focality.
LPR: Since we know that the binary model has the random choice probability $p(a, \{a, b\}) = \frac{u(a)}{u(a)+u(b)}$, the ratio of $r(a, b) = \frac{u(a)}{u(b)}$ for any $a, b \in X$. Thus,

$$r(a, c) = \frac{u(a)}{u(c)} = \frac{u(a)}{u(b)} \times \frac{u(b)}{u(c)} = r(a, b) \cdot r(b, c).$$

Revealed IIA: For any $a, b \in A$, $r_A(a, b) = \frac{u(a)(1+\delta(a)\mathbb{1}\{a \in F(A)\})}{u(b)(1+\delta(b)\mathbb{1}\{b \in F(A)\})}$. If $r_A(c, a) \geq r(c, a)$ for all $c \in A$, we must have $r_A(c, a) = \frac{u(c)(1+\delta(c)\mathbb{1}\{c \in F(A)\})}{u(a)} \geq \frac{u(c)}{u(a)} = r(c, a)$. Thus, for any $a, b \in A$, if $r_A(c, a) \geq r(c, a)$ and $r_A(c, b) \geq r(c, b)$ for all $c \in A$, then $r_A(a, b) = \frac{u(a)}{u(b)} = r(a, b)$.

Menu-Independent Focality: For any $a \in F(A) \cap F(B)$, $b \in N(A)$ and $c \in N(A)$, we can get:

$$\frac{r_A(a, b)}{r(a, b)} = \frac{u(a)(1 + \delta(a))}{u(b)} / \frac{u(a)}{u(b)} = 1 + \delta(a),$$

and

$$\frac{r_B(a, c)}{r(a, c)} = \frac{u(a)(1 + \delta(a))}{u(c)} / \frac{u(a)}{u(c)} = 1 + \delta(a).$$

Thus, $\frac{r_A(a,b)}{r(a,b)} = \frac{r_B(a,c)}{r(a,c)} = 1 + \delta(a)$.

Sufficiency. We will prove the sufficiency by three steps.

Step 1. We first fix some $a^* \in X$. Then we can construct the utility function $u : X \to R_{++}$ in the following way: $u(a) = r(a, a^*)$ for any $a \in X$. Based on the Luce's Product Rule, we know that $r(a, b) = r(a, a^*)r(a^*, b)$, which is $r(a, b) = u(a) \times \frac{1}{u(b)}$ according to the construction of u. Since $\frac{p(a,\{a,b\})}{p(b,\{a,b\})} = \frac{u(a)}{u(b)}$, and $p(a, \{a, b\}) + p(b, \{a, b\}) = 1$, then we can get: $p(a, \{a, b\}) = \frac{u(a)}{u(a)+u(b)}$.
Step 2. For any $A \in \mathscr{A}$ with $|A| \geq 3$, we derive the focal set and non-focal set for A.

Fact 1. There is $F^*(A) \subseteq A$, such that $a \in F^*(A)$ and $b, b' \in N^*(A) = A/F^*(A)$,

$$r_A(b, b') = r(b, b'), \text{ and } r_A(a, b) > r(a, b).$$

Proof of Fact 1: We need to show that for any $b, b' \in N^*(A) = A/F^*(A)$, $r_A(b, b') = r(b, b')$, and for any $a \in F^*(A)$, $b \in N^*(A)$, then $r_A(a, b) > r(a, b)$.
1. For any $b, b' \in N^*(A)$, $r_A(c, b) \geq r(c, b)$ and $r_A(c, b') \geq r(c, b')$ for all $c \in A$ according to the definition (Revealed Focality) $N^*(A)$. By Revealed IIA, $r_A(b, b') = r(b, b')$.
2. For any $b \in N^*(A)$, we have $r_A(c, b) \geq r(c, b)$ for all $c \in A$. For all $b' \in A$ such that $r_A(b', b) = r(b, b')$, we must have

$$r_A(c, b) = r_A(c, b')r_A(b', b) \geq r(c, b')r(b', b) = r(c, b), \text{ by LPR.}$$

Then, we can get $r_A(c, b') \geq r(c, b')$ for all $c \in A$ and all b' such that $r_A(b, b') = r(b, b')$. By the definition of $N^*(A)$, $b' \in N^*(A)$. Then for any $a \in A$ such that $r_A(a, b) > r(a, b)$, $a \in F^*(A)$ by the definition of (Revealed Focality) $F^*(A)$. Thus, we also proved that for any $a \in F^*(A)$ and $b \in N^*(A)$, $r_A(a, b) > r(a, b)$.

Step 3. Let $|F^*(A)|, |N^*(A)| \geq 1$, and $F^*(A) = \{a_1, ..., a_n\}$, $N^*(A) = \{b_1, ..., b_m\}$. By the construction of u, $r_A(b_i, b_1) = r(b_i, a_1) = \frac{u(b_i)}{u(b_1)}$. Thus, $p(b_i, A) = \frac{u(b_i)}{u(b_1)} \times p(b_1, A)$. For any $a_j \in F^*(A)$, we have $r_A(a_j, b_1) > r(a_j, b_1)$. Denote that $\frac{r_A(a_j,b_1)}{r(a_j,b_1)} = 1 + \delta(a_j, A)$. Similarly, if $a_j \in F^*(B)$ for some $B \in \mathscr{A}$ and $b_1^* \in B$, we have: $\frac{r_A(a_j,b_1^*)}{r(a_j,b_1^*)} = 1 + \delta(a_j, B)$. By Menu-Independent Focality, we can further denote $\frac{r_A(a_j,b_1)}{r(a_j,b_1)} = \frac{r_A(a_j,b_1^*)}{r(a_j,b_1^*)} = 1 + \delta(a_j)$.

Then, by LPR, $r_A(a_j, a_1) = r_A(a_j, b_1) \times r_A(b_1, a_1) = \frac{1+\delta(a_j)}{1+\delta(a_1)} \times r(a_j, b_1)r(b_1, a_1) = \frac{1+\delta(a_j)}{1+\delta(a_1)}r(a_j, a_1)$. Since $r(a_j, a_1) = \frac{u(a_j)}{u(a_1)}$, $r_A(a_j, a_1) = \frac{1+\delta(a_j)}{1+\delta(a_1)}\frac{u(a_j)}{u(a_1)}$. Thus, $p(a_j) = \frac{u(a_j)(1+\delta(a_j))}{u(a_1)(1+\delta(a_1))} \times p(a_1, A)$. Then, we have:

$$1 = \sum_{i=1}^{n} p(a_i, A) + \sum_{j=1}^{m} p(b_j, A) = \frac{\sum_{i=1}^{n} u(a_i)(1 + \delta(a_i))}{u(a_1)(1 + \delta(a_1))} \cdot p(a_1, A) + \frac{\sum_{j=1}^{m} u(b_j)}{u(b_1)} \cdot p(b_1, A).$$

Now, we can derive $p(b_1, A)$, which is:

$$\begin{aligned} p(b_1, A) = \frac{p(b_1, A)}{1} &= \frac{p(b_1, A)}{\frac{\sum_{i=1}^n u(a_i)(1+\delta(a_i))}{u(a_1)(1+\delta(a_1))} \cdot p(a_1, A) + \frac{\sum_{j=1}^m u(b_j)}{u(b_1)} \cdot p(b_1, A)}, \\ &= \frac{u(b_1)}{\sum_{i=1}^n u(a_i)(1+\delta(a_i)) \frac{u(b_1)}{u(a_1)(1+\delta(a_1))} \cdot \frac{p(a_1,A)}{p(b_1,A)} + \sum_{j=1}^m u(b_j)} \\ &= \frac{u(b_1)}{\sum_{i=1}^n u(a_i)(1+\delta(a_i)) + \sum_{j=1}^m u(b_j)}, \text{ by the definition of } \delta(a_i). \end{aligned}$$

Since $\sum_{i=1}^n p(a_i, A) + \sum_{j=1}^m p(b_j, A) = 1$, we can also get $p(a_i, A)$, which is:

$$p(a_i, A) = \frac{u(a_i)(1+\delta(a_i))}{\sum_{i=1}^n u(a_i)(1+\delta(a_i)) + \sum_{j=1}^m u(b_j)}.$$

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