Weak Players, Strong Players, and Signed Network Formations*

Eric Bahel[†], Sunjin Kim[‡], and Sudipta Sarangi[§]

Department of Economics, Virginia Tech, Blacksburg, VA 24060

November 23, 2020

Abstract

In a model of signed network formation as proposed by Hiller (2017), this paper studies the possible Nash equilibrium configurations. We characterize the conditions under which complete networks or segregation into two uneven groups can be sustained in the equilibrium in the case of homogeneous agents. We also specify the Nash equilibria in the case of heterogeneous agents. In the model with four agents and two types, we specify all Nash equilibrium network configurations and categorize the equilibria to 3 categories-Utopia network, positive assortative matching, and disassortative matching. Strong (weak) player refers to a player who has a greater (lower) exogenous intrinsic strength. The first Nash equilibrium configuration, Utopia network, obtains when everyone is friend with each other. The second Nash equilibrium configuration, positive assortative matching, is such that players of the same type coalesce. In the third configuration, disassortative matching, some players have friends whose types are different from them. We further generalize each Nash equilibrium configurations to the n–player case; and we derive the specific conditions under which they arise in a Nash equilibrium.

JEL Codes: D74, D85

*The formal title is "Equilibrium configurations in the heterogeneous model of signed network formation."

[†]erbahel@vt.edu

[‡]sunjin@vt.edu

[§]ssarangi@vt.edu

A signed network, which is an application of graph theory, is efficient to describe this interaction between friendship and enmity in the same model. Hence, it is a useful tool to analyze a society where cooperative and noncooperative relationships coexist. The signed network consists of nodes, positive links, and negative links. The nodes denote people, the positive link denotes friendship, and the negative link means conflict. The literature on network economics focuses on positive links between players, which have been well documented in the research on peer effects. Also, economists have studied uncooperative behavior and antagonistic relationships, such as competition and conflicts. However, an interaction between the positive relationship and the negative relationship has not been studied well as much as each relationship.

In history, there are many examples where conflict affects friendship and vice versa. World War II is the most famous instance. In the war, almost all nations of the earth were divided into two parts. Most of these countries were not related to the war at first but became friends or enemies with each other because of indirect relationships. Nazi Germany engaged the U.S.A. after Pear Harbour by Japan. Also, the Soviet Union joined the Allies because Nazi Germany attacked the Soviet Union. As another example, in the 18th century, Europe except for France made an alliance against Napoleon's French Empire even though they were enemies against each other before the war. In the Eastern world, Kuomintang and Communist party collaborated to fight against the Empire of Japan in World War II. However, after World War II, they terminated the collaboration and initiated the Chinese Civil war.

The signed network model was, for the first time, used in Sociology as the name of the signed graph theory to describe a balanced relationship among three entities. Sociologists had the idea that if two people were friends, it is balanced to have the same opinion to the other entity. If they were enemies, it is also balanced to have different opinions. They realized that the signed graph in graph theory was useful to formalize the idea because the signed graph consists of two kinds of links. The theory has been used not only in sociology but in other disciplines such as international relations and politics. The signed network model was, for the first time, used in Sociology as the name of the signed graph theory to describe a balanced relationship among three entities. Sociologists had the idea that if two people were friends, it is balanced to have the same opinion to the other entity. If they were enemies, it is also balanced to have different opinions. They realized that the signed graph in graph theory was useful to formalize the idea because the signed graph consists of two kinds of links. The signed graph theory has been applied not only in sociology but in other disciplines such as the study of international relations (Brown, 1979) and politics (Laumann and Pappi, 1973). In this researches, there had been a common assumption that the balanced signed graph is a kind of equilibrium, and there was a force to enforce a network structurally balanced.

Hiller (2017), for the first time, proposed a model of signed network formation model using a game theoretical approach. In his model, players can either extend a friendly (positive) or an antagonistic (negative) link to each of the remaining agents. His main finding is that every Nash Equilibrium obeys weak structural balance even though there is no direct benefit for each player when their relationship satisfies structural balance in the network. It also means that the Nash equilibriums configurations are such that either all links are positive or players can be divided into distinct sets where people in each set are friends with each other. He further characterizes pure strategy Nash equilibria for a general class of payoff functions which map strength of agents into extraction payoff under the antagonistic relationship.

This paper is an extension of Hiller (2017) regarding the heterogeneous players. We analyze Nash equilibrium configurations when players are homogeneous or have one of two types: strong or weak. This simple heterogeneous model is tractable to analyze. Also, even in this simple comparison, it is possible to detect significant changes. An agent's real strength in conflict is the sum of her intrinsic strength plus the intrinsic strength of her friends or allies. It is named network strength. That is, each player can increase their network strength by making alliances.

One of the main findings we find in this heterogeneous players model is the sufficient condition for which all Nash equilibrium network configurations show assortative matching. Positive assortative matching is a matching between the same type of players. In matching theories, positive assortative matching has been a major topic and researched. For example, it is a stylized fact that there is positive assortative matching in marriage markets by education level. [1]

It means that a highly educated male tends to get married to a highly educated female. On the other hand, some labor economists borrowed this terminology from the marriage market model to labor economics to describe a matching between firms and workers. In labor economics, positive assortative matching means that highly productive firms hire highly productive workers. Different from the marriage market, there is a controversy among the researchers whether positive assortative matching is a general property characterizing any labor market or not. In the signed network formation model consisting of heterogeneous players, positive assortative matching means positive relationships exist only between the same type of players. As the antithesis, naturally, between the different types, there is only a negative relationship. In that different types do not have any cooperative relationships, positive assortative matching can be interpreted as a segregation of the society by inherency such as gender, race, and economic background.

[1] In this field, positive assortative matching is also called positive assortative mating.

In this paper, we specify Nash equilibrium for a signed network formation for homogeneous players as a benchmark model. We characterize conditions under which complete networks or segregation into two uneven groups can be sustained in the equilibrium. Then, we also specify Nash equilibrium in case of binary heterogeneous players. In the homogeneous model, all Nash equilibrium configurations are classified into three categories: Utopia network, positive assortative matching, and disassortative matching. In Utopia network, each player is a friend to every other player. In the first type of equilibrium is when all players are friends to each other regardless of their types. Generally, high level of the conflict cost can make the networks with no conflict in equilibrium. However, there is a striking result that the conflict cost can be lower in equilibrium when the number of strong players increases. This is because the strong players hold each other in check so that the peaceful network is sustainable. In a positive assortative matching, players coalesce only with the same type of players. We categorize positive assortative matching according to who is dominant in the network configuration. If every strong player has higher network strength than every weak player, we call the network strong dominant. On the other hand, if every weak player has higher network strength than every strong player because weak players are the majority in the positive assortative matching, the network is called weak dominant. Lastly, there exist positive assortative matchings neither strong dominant nor weak dominant. Out major finding is that all Nash equilibrium configuration is strong dominant positive assortative matching when the gap between type is huge, and the conflict cost is small. Also, we derive a sufficient condition for which weak dominant positive assortative matching can be a Nash equilibrium. The positive assortative matching gives an intuition for social phenomena such as discrimination, segregation, and bullying. The results can give an intuition when discrimination happens. Lastly, in disassortative matching, there exists at least one friendship link between the different types of players.

In Section 1, we review the relevant literature. In Section 2, we introduce our model and definitions. In Section 3, we show the result of the research. Lastly, We conclude in Section 4.

1 Literature Review

The signed network began to be studied in sociology as the name of the signed graph theory. Heider (1946) proposed the first idea regarding the interaction between positive relationships and negative relationships among people. He proposed a model where there were two people and one entity. He argued that this triangular graph is balanced if they are friends and their opinions to the entity are the same, or if their relationship is antagonistic and their opinions are different. Luce (1950) established a definition of clique. Clique is a subset where there are reciprocated directed links for every pair of nodes of the clique and the clique is not a proper subset of any other set of elements satisfying this property. Following Heider (1946), Cartwright and Harary (1956) and Harary et al. (1953, 1955) formally developed this idea of a balanced graph using the notions of graph theory. They defined a structurally balanced graph with a local property that all triads in the graph are connected either with three positive links or with one positive link and two negative links. Thus, in any structurally balanced graph, my friend's friend is my friend, my enemy's friend is my enemy, and my enemy's enemy is my friend. They also showed that any structurally balanced graph satisfies a global property that the whole graph is segregated to two cliques. Between the two groups, there are only negative links. Davis (1967) defined clustering which had the same meaning to weakly structural balance. Clustering is a partition of the node set into multiple subsets where each positive link connects a node with another point in the same subset and each negative line connects a point with another point in the different subset. When a network has clustering, my friend's friend is always my friend, my enemy's friend is my enemy, but my enemy's enemy is not necessarily my friend. On the other hand, in a structurally balanced network, an enemy of my enemy is my friend.

Recently, some researchers approached the signed network model by using the game-theoretical method. They focused on the endogenous network formation where players are free to choose their relationships with the other players for their interest. Hiller (2017) studied a signed network formation model with positive and negative links. In this model, players could choose friendship and enmities. Their choices formed the network which determined their payoff. In this model, players do not feel any consonance to structural balance or dissonance to structural unbalance. However, any network in equilibrium is structurally balanced. This result answers why we can observe structural balance in the network besides the explanation that people feel consonance to the balanced structure. Jackson and Nei (2015) analyzed another signed network formation model where players can choose positive, negative, and no links. They introduced a production factor such as trade. They showed that trade and high war cost decreased conflict in the network formation game and examined the theoretical result with data.

The signed network formation model is based on the network formation model. Jackson and Wolinsky (1996) defined pairwise stability in an idea that both players involved have to consent to form the positive link between them. Before they invented the condition, Nash equilibrium concept was not enough to derive significant network formation. Suppose two people are not friends. If it is better for both of them to have a new friendship, they will make a consensus to be friends. However, this deviation cannot be examined with Nash equilibrium concept because Nash equilibrium only considers deviations by a single player. Calvó-Armengol and İlkılıç (2009) incorporated the condition of pairwise stability and Nash equilibrium, and defined a pairwise-Nash equilibrium network regarding a simultaneous move game of network formation. In pairwise stable networks, only single links are respectively examined whether it is pairwisely stable or not. On the other hand, in pairwise Nash networks, we also check multiple link deviations made by a single player in a spirit of pairwise stability.

Besides the network formation model based on the game-theoretical approach, the signed network model could be applied from other approaches. König et al. (2017) obtained a Nash equilibrium fighting effort when there are alliance and enmity network given. They implemented the equilibrium result in the empirical analysis with the Second Congo War data. They estimated the effects of the network and predicted an expected impact of policy intervening in the conflict. Antal, Krapivsky and Redner (2006) and Cisneros-Velarde and Bullo (2019) studied network formation process, but still assumed that people prefer structural balance. Antal, Krapivsky and Redner (2006) analyzed the formation dynamically by using an updating rule which pursues a structurally balanced network. On the other hand, Cisneros-Velarde and Bullo (2019) used the Nash equilibrium concept to analyze the signed network. However, the utility function in this model still varied depending on the number of balanced triads.

Besides the signed network model, researchers actively have studied the interaction between cooperative behavior and uncooperative behavior in various ways. Goyal, Heidari and Kearns (2014) analyzed a game where firms compete against each other to take consumers on a social network in an emerging market. They implemented dynamic analysis and equilibrium analysis to specify the conditions for efficiency. Grandjean, Tellone and Vergote (2017) modeled a sequential game where players formed a network in the first stage and had a contest in the second stage. They found that social designer can increase the surplus by making the players more asymmetric when free exit is not allowed. They also showed that a barrier to entry may hurt the total surplus, and the networking can act as the barrier to entry. Bozbay and Vesperoni (2018) axiomatically analyzed Tullock contest success function for all against all contest given a network.

In this paper, we derived many results using the general form of functions to describe pillage occurring on the negative relationships, but also used the normalized contest success function to describe in some parts. The origin of this normalized contest success function is from contest theory. Contest theory is a discipline studying competition dealing with conflict. Tullock (1967) initiated a discussion of Contest. He argued that tariff and monopoly were inefficient because these brought a welfare transfer from the import sector to the domestic production and from consumers to the monopolist, not generating new welfare. Moreover, this welfare transfer triggers a competition to take or keep the transferred welfare. This competition was named rent-seeking activity, later. To describe the conflict, Tullock (1980) introduced a mathematical model. This model is now known as "Tullock contest success function". Hirshleifer (1989) analyzed Tullock contest success function and suggested another contest success function.[2] Hiller (2017) defined normalized contest success functions by subtracting $\frac{1}{2}$ from the contest success functions. This normalized contest success function describes a zero-sum game conflict on the negative relationship. One good thing of this function is that we can observe how a change of zero-sum game conflict by controlling the level of conflict technology. In this function, there is a parameter implying the conflict technology. If the parameter is high, small difference in strength between the players results in a huge difference in this conflict.

Lastly, positive assortative matching has been documented in economics. Roy (1951) mentioned his idea that earning had not been determined independently but had been affected by various factors such that who meets whom as their colleague. Becker (1973) formalized a marriage market model where men tend to marry women with similar traits. Following Becker (1973), many researchers developed the matching theory and studied the condition when matching is positive assortative. (Shimer and Smith, 2000; Legros and Newman, 2007; Johnson, 2013; Li, Sun and Chen, 2013) At the same time, other researchers were interested in whether matching is a really positive assortative in the real world or not. Celikaksoy, Nielsen and Verner (2006), Chiappori, Oreffice and Quintana-Domeque (2012), Greenwood et al. (2014), and Siow (2015) showed that there exists positive assortative matching by their education level in the marriage market. In labor economics, researchers have reported the existence of positive assortative matching between productive firms and productive workers. (Kremer, 1993; Andrews et al., 2012; Mendes, Van Den Berg and Lindeboom, 2010)[3] Lastly, there were trials to draw positive assortative matching using joint liability in a loan market to overcome aymmetric information problem. (Van Tassel, 1999; Ghatak, 1999, 2000)

2 Model

Let $N = \{1, 2, \ldots, n\}$ denote the set of players. We assume that there are two types of players, $t \in \{s, w\}$, where s represents strong type and w represents weak type. Let N_t, $t \in \{s, w\}$, denote the set of players whose cardinality is given by n_t. Thus, $N = N_S \cup N_W$

[2] Hirshleifer (1989) named Tullock contest success function "contest success function in ratio form" and called the new contest success functoin "contest success function in difference form".

[3] There are other papers reporting disassortative matching between firms and workers, ((Abowd et al., 2004; Andrews et al., 2006) but they were not published.

and $n = n_s + n_w$. Based on her own type, each player has her own intrinsic power. So player i has an intrinsic strength θ_s if $i \in N_S$, and θ_w if $i \in N_w$. By definition $\theta_s > \theta_w > 0$.

Every player can either extend a positive (friendly) directed link or a negative (antagonistic) directed link to the remaining players. Player $i \in N$ chooses $g_{ij} \in \{1; -1\}$ for all $j \in N \setminus \{i\}$ where 1 denotes the friendly link and -1 denotes the negative link. Aggregating all choices, player i's strategy is a vector $g_i = (g_{i;1}; g_{i;2}; \ldots; g_{i;i-1}; g_{i;i+1}; \ldots; g_{i;n})$. Each element is represented by a directed link from one player to the other. The space of a player i's strategy g_i is defined by G_i for all $i \in N$. Let $g_{-i} = (g_1; g_2; \ldots; g_{i-1}; g_{i+1}; \ldots; g_n)$ be a set of all the players' strategies except for player i's strategy g_i. Therefore, $(g_i; g_{-i})$ is the same to g. The players' strategy profile is a directed network $g = (g_1; g_2; \ldots; g_n)$. The joint strategy space for g is given by $G = G_1 \times \ldots \times G_n$. To express a deviation strategy, a change in directed link $g_{i;j}$ is denoted as follows: Given a network g, $g + g^+_{i;j}$ changes the directed link from $g_{i;j} = -1$ to $g_{i;j} = 1$. Similarly, $g + g^-_{i;j}$ changes the directed link from $g_{i;j} = 1$ to $g_{i;j} = -1$. To denote player i's deviation strategy and its strategy profile, we normally use notations g^0_i and $g^0 = g + \sum_{j \in A} g^+_{i;j} + \sum_{j \in B} g^-_{i;j}$ for $A = \{g_{i;j} | g_{i;j} = 1\}$ and $B = \{g_{i;j} | g_{i;j} = -1\}$.

Relationships between players are formed according to their attitude towards each other. The relationship between i and j is denoted by an undirected link $\bar{g}_{ij} = \bar{g}_{j;i} = \min\{g_{i;j}; g_{j;i}\} \in \{-1; 1\}$. As this min function implies, the worst attitude between them determines this bilateral relationship. If both players are friendly, then they will be good friends. If one of them is antagonistic, then they will have the undirected negative link. Similar to the directed network g, player i's relationships are represented by $\bar{g}_i = (\bar{g}_{i;1}; \bar{g}_{i;2}; \ldots; \bar{g}_{i;i-1}; \bar{g}_{i;i+1}; \ldots; \bar{g}_{i;n}) \in \bar{G}_i$, and the undirected network is $\bar{g} = (\bar{g}_1; \bar{g}_2; \ldots; \bar{g}_n) \in \bar{G}$.

If $\bar{g}_{ij} = 1$, player i and j are friends. These two players can strengthen each other's power. Let $N^+_i(g) = \{j \in N | \bar{g}_{ij} = 1\}$ denote a set of players with whom player i has an undirected positive link, and y_i denote player i's network strength, which is a result of her intrinsic quality and her network of friends. Thus, y_i is determined as follows.

$$y_i(g_i; g_{-i}) = \theta_i + \sum_{j \in N^+_i(g)} \theta_j :$$

Therefore, apart from the type of players, each player can have different level of y_i depending on g. An important consequence of this is that a weak player can end up with high strength simply by having strong allies. To sort the players by the network strength, let $y_1 < y_2 < \ldots < y_m$ be the network strengths in a network g where $1 \le m \le n$. Then, let $P_i \in \{P_1; P_2; \ldots; P_m\}$ denote the set of the players whose network strengths are y_i. That is, $P_1 (P_m)$ is the set of the weakest (strongest) players.

If $\bar{g}_{ij} = -1$, player i and j are enemies against each other. They do not share their strength and enter a zero-sum competition. $N_i^e(g) = \{j \in N | g_{i;j} = \;\, 1\}$ denotes the set of players to whom player i extends a negative link. $N_i(g) = \{j \in N | g_{i;j} = \;\, 1\}$ denotes the set of players with whom i has a negative link in the undirected network g, i.e., the set of players with whom player i is engaged in conflict. In this zero-sum game, the player with the higher network strength extracts payoff from the player with lower network strength. Let $f(y_i(g); y_j(g))$ denote player i's extraction from player j in a network g. The network g determines y_i and y_j, which determines $f(y_i(g); y_j(g))$. Thus, the extraction function $f(y_i(g); y_j(g))$ is ultimately a function of g, but it is also possible to define some property of the extractions function regarding the value of y_i and y_j. We will abbreviate the extraction function to $f(y_i; y_j)$ when there is no confusion.

Following Hiller (2017), we define the general extraction function as follows. Given players, for any g, if $f(y_i(g); y_j(g))$ has a lower bound, $\underline{f}(g) = \inf(f(y_i(g); y_j(g))$. In the same way, given n players, for any g, if $f(y_i(g); y_j(g))$ has an upper bound, $f(g) = \text{Sup}(f(y_i(g); y_j(g))$. When $y_i > y_j$ the value of $f(y_i; y_j)$ is positive, but when $y_i < y_j$, it is negative. Moreover, if a pair of players with equal strength is negatively connected, then no payoff extraction takes place: $f(y_i; y_j) = 0 \;\; \forall y_i; y_j : y_i = y_j$. Next, we assume that $f(y_i; y_j)$ is strictly increasing in y_i ($\frac{df(y_i;y_j)}{dy_i} > 0$) and decreasing in y_j ($\frac{df(y_i;y_j)}{dy_j} < 0$). Following the definition of zero-sum game, the value a player extracts through an antagonistic link is assumed to be another player's loss, hence $f(y_i; y_j) + f(y_j; y_i) = 0; \;\forall y_i; y_j$.

We further make the following assumption: (i) $f(y_i; y_j)$ is homogeneous of degree 0. That is, $f(y_i; y_j) = f(\frac{y_i}{y_j}; 1)$. Therefore, this function can be used to describe conflicts where the relative size of strength matters. Using this property, let's denote $a = \frac{y_i}{y_j}$, a ratio of the different intrinsic strengths. Then $f(y_i; y_j) = f(a; 1)$. (ii) $\frac{d^2 f(y_i;y_j)}{dy_i^2} < 0$ when $y_i > y_j$. It means that a marginal return of extraction in network strength is decreasing.

Following Hiller (2017), we also consider a normalized contest success function which satisfies all the properties of $f(y_i; y_j)$ mentioned above. This is shown in Appendix.

Definition 1 (Hiller (2017)) The normalized contest success function (in ratio form) is

$$h(y_i; y_j; \;) = \frac{y_i}{y_i + y_j} \quad \frac{1}{2} \in (\;\frac{1}{2}; \frac{1}{2})$$

where $\; > 0$.

This normalized contest success function is a special case of extraction function which incorporates an interesting parameter. This parameter characterizes the technology of extraction, where $\frac{dh}{d} > 0$ when $y_i > y_j$ This property implies that as increases, $h(y_i; y_j; \;)$ increases for given y_i and y_j. This allow us to examine some interesting cases. When is high, small differences can determines the winner of the conflict. When is low, i.e., when the two players have similar extraction technologies or are evenly matched, for any value of y_i and y_j, the value of the normalized contest success function is close to zero. Formally, if goes to infinity, $h(y_i, y_j;)$ is close to $\frac{1}{2}$ when $y_i > y_j$, and if goes to 0, $h(y_i, y_j;)$ is close to 0. This is because $h(y_i, y_j;) \in (\frac{1}{2}, \frac{1}{2})$ for any y_i, y_j and , and $f(y_i, y_j) = 0$ when $y_i = y_j$. The proof of the property is also in Appendix.

In this model, there is no cost associated with friendship, but there is a cost associated with conflict. Extending a negative link results is akin to picking a fight, which imposes a cost $\varepsilon > 0$. When there is at least one directed negative link between players, both players are assumed to engage in the bilateral conflict, and the cost of conflict is 0.[4] Note that the same cost value may either exclude the possibility of conflict or not depending on the parameters of networks such as the number of players. For example, for the homogeneous case, when there are five players, $f(g) = f(4, 1)$; and if there are six players, $f(g) = f(5, 1)$. If $f(4, 1) < \varepsilon + < f(5, 1)$, this $\varepsilon +$ excludes the possibility of conflict in a network with five players, but it allows for conflict in a network with six players. Our research considers this aspect by relaxing the restriction on $\varepsilon +$.[5]

The utility of player i under a strategy profile g is given by

$$u_i(g_i, g_{-i}) = \sum_{j \in N_i(g_i)} f(y_i(g_i, g_{-i}), y_j(g_j, g_{-j})) \quad |N_i^e(g)|\varepsilon \quad |N_i(g)| : \qquad (1)$$

As mentioned above, we also use the normalized contest success function $h(y_i, y_j;)$ instead of $f(y_i, y_j)$. Let $u_i^h(g_i, g_{-i};)$ the utility of a player i under a strategy profile g with $h(y_i, y_j;)$.

$$u_i^h(g_i, g_{-i};) = \sum_{j \in N_i(g_i)} h(y_i(g_i, g_{-i}), y_j(g_j, g_{-j});) \quad |N_i^e(g)|\varepsilon \quad |N_i(g)| : \qquad (1.1)$$

In Section 3, we use $f(y_i, y_j)$ and $u_i(g)$ in general, but sometimes use $h(y_i, y_j;)$ and $u_i^h(g)$ to study the change related to . If results are about $h(y_i, y_j;)$ and $u_i^h(g)$, we specify that h is used instead of **f** in each result. Otherwise, the result holds for f as well as h.

We study only pure strategy Nash equilibria which is defined in the usual way. A strategy g is a Nash equilibrium if and only if

$$u_i(g_i, g_{-i}) \quad u_i(g_i, g_{-i}) \; \forall g_i \in G_i, \; \forall i \in N:$$

[4] The cost can be interpreted as an opportunity cost from a loss of complementarity.

[5] Hiller (2017) assumed that $\varepsilon + < f(\sum_{j \in N \{k\}} {}_j, {}_k)$ where ${}_k \quad {}_j \; \forall j \in N$. It means that the cost of conflict in his model is always lower than the maximum possible extraction. He focused on conflict phenomenon by using the assumption and did not discuss the case where the networks with no conflict are the Nash equilibrium. Our model relaxes the assumption, so $\varepsilon +$ is only required to be larger than 0.

(a) Not structurally balanced

(b) Structurally balanced

(c) Weakly structurally balanced

Figure 1: Networks with di erent properties

2.1 Structurally balanced networks

Equilibrium networks share some common properties. First, every Nash equilibrium strategy g shows clustering. Cartwright and Harary (1956) introduced the term \clique" from graph theory to this literature. [6]

They rede ned clique as a subset of nodes (players) within which all links are positive and across which they are negative. They also rede ned structurally balanced graphs (networks) in the context of a signed graph as follows.

De nition 2 (Cartwright and Harary, 1956, p. 286, Structure theorem) A signed graph is balanced if and only if its nodes can be separated into two mutually exclusive cliques such that each positive undirected link joins two players within the same clique and each negative undirected link joins players between di erent cliques.

Davis (1967) relaxed the condition the de nition in terms of cliques He suggested calling the multiple cliques' phenomenon clustering. By borrowing terms from Cartwright and Harary (1956), a formal de nition can be written as follows.

De nition 3 A signed graph has a clustering if and only if its nodes can be separated into multiple mutually exclusive cliques such that each positive undirected link joins two players within the same clique and each negative undirected link joins players between di erent cliques.

A signed network which has a clustering is also called a weakly balanced signed network.

[6] In the graph theory, the clique is de ned as a subset of nodes which are fully connected but not a subset of another subset satisfying this property. For more detail, see Luce (1950)

Figure 2: Categories in weakly balanced signed networks

2.2 Types of network configurations

Before going to the next section for analysis, let us define some significant signed network configurations that arise with two types of players. Note that there are only four categories of the weakly balanced signed networks with the heterogeneous players.

Definition 4 Utopia network is the strategy profile g where $g_{ij} = 1 \ \forall \{i; j\} \in N \times N \setminus \{i\}$.

Utopia network is a peaceful and ideal situation without any conflicts. However, these friendships can only exist between similar agents in some networks.

Definition 5 Positive assortative matching (PM) is a network configuration g where there exists $g_{ij} = 1$ for $\{i; j\} \in N_s \times N_s \setminus \{i\}$ or for $\{i; j\} \in N_w \times N_w \setminus \{i\}$, and $g_{ij} = -1$ $\forall \{i; j\} \in N_s \times N_w$.

We further classify positive assortative matching by order of the network strength related to the type of players. There are three subcategories in this positive assortative matching. In the first category, each strong player has a higher network strength than every weak player. In the second category, each weak player has higher network strength than all of the strong players. This network configuration, which is called the tyranny of the weak, can be formed when the number of weak players is more than that of the strong players. Lastly, the rest of the cases is in the third category. Here is a formal definition for the first and second categories.

Definition 6 i) Strong dominant positive assortative matching (SPM) is a positive assortative matching where $y_i > y_j$ for every $i \in N_s$ and $j \in N_w$. ii) Weak dominant positive assortative matching (WPM) is a positive assortative matching where $y_j > y_i$ for every $i \in N_s$ and $j \in N_w$.

While there is no dominant type in the third category, the strong/weak type is dominant in strong/weak dominant positive assortative matching. It implies that there is a clear direction of extraction between the types. Therefore, when a network is either strong or weak dominant positive assortative matching, people can feel that there is structural inequality by their attribute in their society.

As extreme cases of these dominant positive assortative matchings, there are network configurations where all of the same type players are fully connected with the positive links, and the negative links only exist between the different types of players. In other words, the same type players compose one complete network with the positive links. Because there are two types of players in this model, there exist two complete friendship networks. Let's call these networks complete strong dominant positive assortative matching (CSPM) and complete weak dominant positive assortative matching (CWPM).

Definition 7 i) Complete strong dominant positive assortative matching (CSPM) is a strong positive assortative matching such that $g_{i,j} = 1 \ \forall \ i,j \in N_s$ and $i,j \in N_w$.

ii) Complete weak dominant positive assortative matching (CWPM) is a weak positive assortative matching such that $g_{i,j} = 1 \ \forall \ i,j \in N_s$ and $i,j \in N_w$.

Besides Utopia network and positive assortative matching, there are network configurations where the different types of players are friends, but there exist negative relationships, too. Let's call this family of networks disassortative matching.[7]

Definition 8 Disassortative matching is a network configuration such that

(i) there exists $g_{i,j} = 1$ for $i \in N_s$ $j \in N_w$, and

(ii) there exists $g_{i,j} = -1$ for any $i,j \in N$.

Lastly, there is a configuration where only negative links exist. It can be called a war of all against all. In this study, I do not consider this category of networks. When there are two types of players, if there are more than two players, there are always two players whose network strengths are the same in this configuration. They have a negative link between them by definition of the war of all against all. Then, the players whose network strengths are the same does not have an incentive to maintain the negative link. Also, it violates Remark 1, which will be mentioned in the first part of the section of Analysis.

[7] In other literature, it is also called negative assortative matching.

Figure 3: If a network contains the above triad with two positive links and one negative link, it violates Remark 1 and the definition of a clustered signed network in Definition 3.

3 Analysis

Hiller (2017) showed that every network configuration has a clustering in equilibrium.

Remark 1 (Hiller, 2017, p.1066, Proposition 1) In any Nash equilibrium, if $y_i(g) = y_j(g)$, then $g_{i,j} = 1$, and if $y_i(g) \neq y_j(g)$, then $g_{i,j} = -1$.

If a network does not consist of cliques, there is a triad i, j, and k where $g_{i,j} = g_{j,k} = 1$ buy $g_{i,k} = -1$. Then, it violates the statement that if $y_i(g) = y_j(g)$, then $g_{i,j} = 1$, and if $y_i(g) \neq y_j(g)$, then $g_{i,j} = -1$. Hence, any network with this triad cannot be a Nash equilibrium.

Remark 1 is a necessary condition for any network g to be a Nash equilibrium. The formal proof is in his paper, but we shortly show the logic of the proof. Regardless of the types, if $y_i(g) = y_j(g)$ but $g_{i,j} = -1$ in a g, $f(y_i(g); y_j(g)) = 0$ but $\varepsilon + \; > 0$. Therefore, either i or j want to terminate this futile conflict. Regarding the second statement, it is possible to prove it indirectly by showing that every player in the same $P_a(g)$ has the same strategy to the other players out of $P_a(g)$. For example, suppose i and j in $P_m(g)$ has nonidentical strategies to the other players. Without loss of generality, if $u_i(g) < u_j(g)$, then i imitates j's strategy in her deviation g'_i. Then it increases i's utility. Roughly speaking, suppose $g_{i,k} = 1$ and $g_{j,k} = -1$ for $k \in P_{m-1}$. If player i imitates j's strategy so $g'_{i,k} = -1$, it decreases k's network strength. Therefore, $u_j(g') > u_j(g)$. $u_i(g') = u_j(g')$; so $u_i(g') > u_i(g)$. This example is just for the players in $P_m(g)$, but it is possible to repeat this way for the other relationships between $P_a(g)$. Note that this logic is not relevant of the type of each player. Therefore, Remark 1 is useful to analyze the heterogeneous player model.

With Remark 1 in the endogenous signed network formation game, first, we can only consider weak balanced networks in equilibrium. Note that players in the same clique have the same network strength y_i. It also simplifies the study of this endogenous network formation model. Second, if players in different cliques have the same network strength y_i, then g is not a Nash equilibrium. This property excludes some weakly balanced network configurations from Nash equilibria before checking each player's incentive to deviate from the networks. Third, in every Nash equilibrium, players in the same clique have the same network strength. If a pair of players have different network strength although they are in the same clique, it contradicts Proposition 1. As mentioned above, P_i denotes a set which ranks the network strength of the players. Therefore, in equilibrium, P_i can represent each clique according to the rank of network strength which the players in the clique have.

On the other hand, there is a deterministic relationship with a negative undirected link and negative directed links in equilibrium. In the model, if there is at least one negative directed link in a bilateral relationship, then the undirected link between the players has a negative sign. However, in equilibrium, a player with stronger network strength extends negative links to another player with weaker network strength. On the other hand, the player with weaker network strength extends positive links to the player with stronger network strength in equilibrium. Hiller (2017) mentioned the result as given below:

Remark 2 (Hiller, 2017, p.1073, Lemma 1 and 2)

1. In any Nash equilibrium, there does not exist a pair of agents i and j such that $g_{i;j} = g_{j;i} = 1$.

2. In any Nash equilibrium, if $g_{i;j} = 1$ with $y_i(g) < y_j(g)$, then $g_{i;j} = 1$.

For every bilateral relationship, a negative relationship is only profitable when a player has a higher network strength. Thus, the player with the higher network strength extends the negative link with the cost '' for picking a fight. However, the player who has the lower network strength does not want to pay the conflict cost. Thus, he would avoid '' by extending the positive link. Like Remark 1, in this logic of Remark 2, there is no mention of the players' type and their intrinsic strengths. Therefore, Remark 2 is also valid in the heterogeneous player model. Hence, in the following analysis, when we analyze networks in equilibrium, $g_{ij} = g_{ji} = 1$ for each $g_{ij} = 1$, and $g_{ij} = 1$ and $g_{ji} = 1$ for each $g_{ij} = 1$ where $y_i(g) > y_j(g)$ by the definition of positive undirected links and Lemma 2.

3.1 The model with homogeneous players

Let's assume that players are homogenous in the model. The assumption from the benchmark model will be generalized in the latter section. In the following sections, we categorize networks according to the number of cliques. In each category, we summarize Hiller (2017)'s results and generalize some of them.

3.1.1 Networks consisting of one clique: Utopia networks

Hiller (2017) mentioned that the Utopia network is the unique Nash equilibrium if $\varepsilon + \delta$ is sufficiently large.[8] He also showed that if the conflict cost $\varepsilon + \delta$ is appropriately small,[9] then the Utopia network could be a Nash equilibrium. Therefore, Utopia network is always in equilibrium in the homogeneous model.

Remark 3 Let g be Utopia network where $\theta_i = \theta \; \forall i \in N$.

(a) If $\varepsilon + \delta > f(n-1;1)$, then g is the unique Nash equilibrium.

(b) If $\varepsilon + \delta \geq f(n-1;1)$, then g is a Nash equilibrium. (Hiller, 2017, p. 1066, Proposition 2)

In the homogeneous model, $\underline{c}(g) = f(n-1;1)$. If $f(n-1;1) \geq \varepsilon + \delta$, then the bully network, where $n-1$ players extend the undirected negative links to the other player, is a Nash equilibrium because this extraction is profitable. If n increases, then $f(n-1;1)$ also increases. It means that for the same level of conflict cost $\varepsilon + \delta$, if there are few people, Utopia network can be the unique Nash equilibrium. However, if there are more people, then Utopia network is not the unique Nash equilibrium more, but also conflict can occur in equilibrium.

As mentioned in the model, the normalized contest success function $h(n_i;n_j;\theta)$ can be used for $f(n_i;n_j)$. Here, it is possible to characterize a change in the level of conflict cost with respect to the number of players and technology of extraction in equilibrium. Given n and θ, $h(n-1;1;\theta) = \frac{(n-1)^{\theta}}{(n-1)^{\theta}+1} - \frac{1}{2}$ is the minimum of $\varepsilon + \delta$ for the Utopia network to be the unique Nash equilibrium. If $\varepsilon + \delta$ is larger than this threshold, peace is always the rational choice for every player. However, if $\varepsilon + \delta$ is less than this threshold, initiating conflict can be a rational choice, too. Let $\underline{c}^{u}(n)$ denote the minimum of the conflict cost $\varepsilon + \delta$ for the Utopia network to be the unique Nash equilibrium given n with the function $f(y_i;y_j)$. Also, let $\underline{c}^{uh}(n;\theta)$ denote that given n and θ with the function $h(y_i;y_j;\theta)$.

Corollary 1

(i) As n increases, $\underline{c}^{u}(n)$ increases.

(ii) If f is h, as θ increases, $\underline{c}^{uh}$ increases.

Proof.

(i) $\underline{c}^{u}(n) = f(n-1;1)$. $f(n-1;1)$ increases as $n-1$ increases as defined. (ii) $\underline{c}^{uh}(n;\theta) = \frac{(n-1)^{\theta}}{(n-1)^{\theta}+1} - \frac{1}{2}$. The first derivatives are $\frac{\partial \underline{c}^{uh}}{\partial \theta} = \frac{(n-1)^{\theta}\log(n-1)}{\left((n-1)^{\theta}+1\right)^{2}} > 0$. ■

[8] $\varepsilon + \delta \geq f(\sum_{j \in N \setminus \{k\}} \theta_j; \theta_k)$ where $\theta_k \geq \theta_j \; \forall j \in N$.

[9] $\varepsilon + \delta < f(\sum_{j \in N \setminus \{k\}} \theta_j; \theta_k)$ where $\theta_k \geq \theta_j \; \forall j \in N$.

Corollary 1 indicates that when there are small number of players and low level of extraction technology, the Utopia network is unique in equilibrium. As the number of players gets increase and the level of technology is developed, there should be a high level of cost for conflict for the Utopia network to be the unique Nash equilibrium. Therefore, for the same level of the conflict cost, the utopia network can be the unique equilibrium or one of the Nash equilibria according to the number of players and the level of extraction technology.

3.1.2 Networks consisting of two cliques

This section considers the network consisting of two cliques. Hiller (2017) showed that there exists a network configuration in equilibrium such that $n-1$ players form a clique and extend negative links to the other player. Let us name this strategy profile bullying strategy. Bullying strategy is a Nash equilibrium under a condition: $\varepsilon + \delta < f(n-1;1)$. Now, let us generalize his result to the case that there are more than one bullied players. Let C_1 denote the set of players who bully and C_2 denote the set of players who are bullied, where $n_1 = |C_1|$ and $n_2 = |C_2|$. Also, let g^{2C} denote the generalized bullying strategy profile. In g^{2C}, $g_{ij} = -1$ if $i \in C_1$ and $j \in C_2$, and $g_{ij} = 1$ otherwise. Lemma 1 gives conditions under which these network configurations with two cliques can be sustained in a Nash equilibrium.

Lemma 1 Suppose $\theta_i = \theta \ \forall i \in N$. g^{2C} is a Nash equilibrium if and only if $\varepsilon + \delta \geq f(n_1+1;n_2) - n_2(f(n_1+1;n_2) - f(n_1;n_2))$.

The condition $f(n_1+1;n_2) - n_2((f(n_1+1;n_2) - f(n_1;n_2))$ is derived from $u_i(g^{2C}) \geq u_i(g^{2C} + g^+_{ij})$ where $i \in C_1$ and $j \in C_2$. When the above condition is satisfied, then the players in C_1 do not have an incentive to deviate from g^{2C} by extending one positive link. Also, when this condition is satisfied, the other deviations extending multiple links are also non-profitable by the property of decreasing marginal extraction to network strength. The weak players cannot change the relationship between the cliques, so it is enough to check the strong players' incentive. The more detailed proof is in Appendix.

Based on Lemma 1, it is possible to argue that there is a minimum size of C_1 for g to be a Nash equilibrium. Let's define the minimum size $\bar{n}_1$, given n and $\varepsilon + \delta$. If the number of players in C_1 is larger than this minimum size value, then this configuration is in equilibrium.

Proposition 1 Suppose $\theta_i = \theta \ \forall i \in N$. For $\varepsilon + \delta \leq f(n-1;1)$, there exists $\bar{n}_1(n; \varepsilon + \delta) < n$ such that every network configurations with two cliques is a Nash equilibrium if and only if $n_1 \geq \bar{n}_1$.

Example 1 Suppose there are ten players and $f = h(y_i; y_j; 1)$. When $n_1 = 6$, g^{2C} is not a Nash equilibrium for $\varepsilon + \delta > 0$. When $n_1 = 7$, g^{2C} is a Nash equilibrium if $\varepsilon + \delta \geq \frac{16}{110}$. When

$n_1 = 8$, g^{2C} is a Nash equilibrium if" + $\frac{31}{110}$. When $n_1 = 9$, g^{2C} is a Nash equilibrium if " + $\frac{4}{10}$. Lastly, Utopia network is the unique Nash equilibrium if" + > $\frac{4}{10}$.

Once again, this proposition is a generalized version of Hiller (2017)'s Proposition 2. In his proposition, he showed that when there are $n - 1$ players in C_1, this configuration was always a Nash equilibrium. His proposition was proved without the assumptions that $f(y_i; y_j)$ is homogeneous of degree 0, and the marginal return is decreasing in y_i. With these assumptions, this Proposition 1 shows that other configurations are can be Nash equilibria.

Note that all structurally balanced network configurations consist of two cliques. If there are more than two cliques in the network,[10] it is not structurally balanced by definition. Therefore, it is the characterization of Nash equilibrium for any structurally balanced network formation when the players are homogeneous.

3.2 The model with two types of players

Now, let us introduce heterogeneity within the player set. As mentioned in Section 2, each player is either strong or weak, with the assumption that there exists at least one player of each type.

Example: The four players model with two types

In the last section, we have checked all possible network configuration in equilibrium with four homogeneous players and the conditions for each one to be a Nash equilibrium. In this example, we also find all configurations which can be in equilibrium.

As mentioned in the model section, there are three categories of network configurations which can be Nash equilibria: Utopia, positive assortative matching, and disassortative matching. First, there are three Utopia networks in this four-player model. These three networks have a different number of strong (weak) players: (i) $n_s = 3$; $n_w = 1$, (ii) $n_s = 1$; $n_w = 3$, and (iii) $n_s = 2$; $n_w = 2$. Second, there are six positive assortative matchings in equilibrium. Five of them are strong dominant ones, and the other one is weak dominant. In the following list, players in a higher-numbered clique have higher network strength than players in a lower numbered clique. When $N_s = \{1; 2; 3\}$ and $N_w = \{4\}$, there are networks possible to be equilibria such that (i) $C_1 = \{1; 2; 3\}$, $C_2 = \{4\}$, and (ii) $C_1 = \{1; 2\}$, $C_2 = \{3\}$, $C_3 = \{4\}$. When $N_s = \{1\}$ and $N_w = \{2; 3; 4\}$, there are networks possible to be equilibria such that (iii) $C_1 = \{1\}$, $C_2 = \{2; 3; 4\}$, (iv) $C_1 = \{2; 3; 4\}$, $C_2 = \{1\}$ (the only weak dominant positive assortative matching), and (v) $C_1 = \{1\}$, $C_2 = \{2; 3\}$, $C_3 = \{4\}$.

[10] This configuration is called a weakly balanced network.

Figure 4: Utopia networks

When $N_s = \{1;2;3\}$ and $N_w = \{4\}$, there is a network possible to be an equilibrium such that (vi) $C_1 = \{1;2\}$, $C_2 = \{3;4\}$.

Lastly, there are four positive assortative matching in equilibrium. When $N_s = \{1;2;3\}$ and $N_w = \{4\}$, there is a network possible to be an equilibrium such that (i) $C_1 = \{1;2;4\}$, $C_2 = \{3\}$. When $N_s = \{1\}$ and $N_w = \{2;3;4\}$, there is a network possible to be an equilibrium such that (ii) $C_1 = \{1;2;3\}$, $C_2 = \{4\}$. When $N_s = \{1;2;3\}$ and $N_w = \{4\}$, there are networks possible to be equilibria such that (iii) $C_1 = \{1;3;4;\}$, $C_2 = \{2\}$, and (iv) $C_1 = \{1;2;3\}$, $C_2 = \{4\}$.

We draw all figures of possible Nash equilibrium network configurations in Table 4 in Appendix.

Except for these configurations, there are no possible Nash equilibrium configurations in the model with four heterogeneous players. In both this example with four players and the general n players case, these configurations can be classified to three categories: Utopia networks, positive assortative matching, and disassortative matching. we specify the condition for each configuration in Table 4 to be a Nash equilibrium in Appendix. Furthermore, in this section, we generalize these conditions to general case with n players.

3.2.1 Utopia networks

Given the number of strong players and weak players and the ratio of intrinsic strength $a = \frac{s}{w}$, there is a condition of the conflict costs $\epsilon +$ for Utopia network to be a Nash equilibrium. There are some specific deviations which always give higher payoff than the other deviations. Therefore, if Utopia strategy gives more payoff than these particular deviations, then the Utopia strategy profile is a Nash equilibrium. Furthermore, Utopia network is the unique equilibrium when benefit from extending any negative links is always less than the conflict cost.

Theorem 1 Suppose there are n_s strong players and n_w weak players. (i) Utopia network is a Nash equilibrium if and only if $'' +$ $f^e = f(n_s\ {}_s + (n_w\ 1)\ {}_w; (n_s\ 1)\ {}_s + n_w\ {}_w)$. (ii) Utopia network is the unique Nash equilibrium if and only if $'' + \ > f^u = f(n_s\ {}_s + (n_w\ 1)\ {}_w;\ {}_w)$. [11]

Theorem 1 determines the minimum of the con ict cost for Utopia network to be a Nash equilibrium and the other minimum con ict cost for Utopia network to be the unique Nash equilibrium. Given n_s strong players and n_w weak players, f^e is the extraction value when a strong player suddenly attacks another weak player in Utopia network. Even though this strong player initiates the con ict with the weak player, the other players (n_s 1 strong players and n_w 1 weak players) are still friends with the strong player and the weak player. On the other hand, when a weak player with no friend is bullied by one of the other players, each bullying player extracts f^u from the lonely weak player. Note that $f(g) = f^u$. Therefore, if the con ict cost $'' +$ is higher than f^u, then any possible con ict given the set of players is unpro table. Hence, Utopia network where there are no con icts becomes the unique Nash equilibrium.

It is possible to implement comparative statics analysis with the conditions in Theorem 1.

Corollary 2

(i) $\frac{@f^e}{@a} > 0$, $\frac{@f^e}{@n_s} < 0$, and $\frac{@f^e}{@n_w} < 0$.

Also, $\frac{@f^u}{@a} > 0$, $\frac{@f^u}{@n_s} > 0$, and $\frac{@f^u}{@n_w} > 0$.

(ii) Let $n_w = n \quad n_s$. For any given n, $\frac{df^e}{dn_s} < 0$ and $\frac{df^u}{dn_s} > 0$.

Given the di erence ${}_s$ and ${}_w$, if either n_s or n_w increases, f^e decreases. It implies that Utopia network is stable when there are lots of players. Although a sudden con ict happens between players, the weak player can still have support from the rest of the players. It functions as counterchecking a new con ict. Di erent from f^e, f^u is increasing when either n_s or n_w increases. f^u is the possible maximum extraction value, and it increases when the number of players gets larger. Therefore, the level of con ict cost to block non-Utopia network increases as the number of players increases. Collectively, in the large society, Utopia network can be stable if it is attained once, but there are also many other possible networks with con icts in equilibrium.

On the other hand, given the number of players, as $a = \frac{s}{w}$ increases, f^e and f^u increase in the same direction. It implies that as the disparity between the di erent types grows, Utopia network gets unstable and is also hard to be the unique equilibrium.

[11] $f^u > f^e$

Meanwhile, as shown in Corollary 2 (ii), xing the total number of players, it is possible to observe the direction of change in f^e or f^u regarding the ratio of the number of strong players to the number of weak players. If there are more strong players (and the number of weak players decreases), then f^e decreases, but f^u increases. First, regarding f^e, that n_s increases means there are more strong colleagues helping each weak player. Even though a strong player is more durable than each weak player, if every other strong player continues to support the weak player, then it is hard for the strong player to bully the weak player. Thus, without the possibility of collective action, each strong player has to consider the existence of the other strong players. Second, if there are many strong players and if they bully a weak player having no friends together, this attack must be e ective. The e ect would increase as the number of strong players rises. As a result, the cost level to make this attack unpro table increases as the number of strong players increases. Lastly, an increase of n_w has the opposite e ect because it means decreasing n_s.

Additionally, by applying the normalized contest function, it is possible to observe variations of the conditions with respect to . The following corollary is directly derived from Theorem 1 by using the normalized contest success function with $= 1$.

Corollary 3 When $f = h$ and $= 1$, (i) Utopia network is a Nash equilibrium if and only if

$$" + \qquad \frac{a \quad 1}{2((2n_s \quad 1)a + 2n_w \quad 1)}$$

It is satis ed if

$$" + \quad < \frac{1}{4n \quad 2} \text{ and } a \quad \frac{4(" + \quad)n_w + 2(" + \quad) \quad 1}{4(" + \quad)n_s \quad 2(" + \quad) \quad 1}$$

or if

$$" + \qquad \frac{1}{4n \quad 2} \; 8a:$$

(ii) Utopia network is the unique Nash equilibrium if and only if

$$\frac{n_w + n_s \quad 2}{2n_w + 2n_s} < " + \quad < \frac{1}{2} \text{ and } a \quad \frac{2(" + \quad)n_w + n_w \quad 2}{2(" + \quad)n_s \quad n_s}$$

or

$$" + \qquad \frac{1}{2} \; 8a$$

We omit the proof because the conditions are directly derived from Theorem 1.

How about cases when is large or small? In simulations, the minimum boundary for the con ict costs " + increases as increases. In Appendix, there is a graphical result of the simulations. Also, for the extreme cases of, it is possible to derive conditions for the Utopia network to be a Nash equilibrium or the unique Nash equilibrium.

Corollary 4 When $f = h$ and ! 1 , if and only if " + $\frac{1}{2}$, Utopia network is a Nash equilibrium and it is unique.

Corollary 5 When $f = h$ and ! 0, for any given " + > 0, Utopia network is a Nash equilibrium and it is unique.

Corollary 3 con rms the implications from Proposition 1. In the condition for existence, the lower bound of " + increases as rises. Also, in the condition for uniqueness, if n_w or n_s is large, $\frac{n_w + n_s \quad 2}{2n_w + 2n_s}$ gets closer to $\frac{1}{2}$. Corollary 4 and 5 discuss the extreme cases of. When the technology of extraction is highly e cient, then only high con ict cost can intervene in the existence of con ict at the equilibrium. However, if the technology is not developed, then a low con ict cost can prevent the con icts and attain the Utopia network in the equilibrium.

The Utopia networks is a special case consisting of only the positive links. Except for Utopia network, there is at least one negative link in every network con gurations. Contrary to the Utopia network, a con guration with only negative links, which can be called \the war of all against all" cannot be a Nash equilibrium except when there are only one strong player and one weak player. If there are at least three players in these two types heterogeneous player model, at least two players have the same type and have the same aggregate strength. This is a contradiction to Hiller's Proposition 1.

3.2.2 Positive assortative matching

In these con gurations, friendship (or alliances) is only available between the same type of players. However, negative relationships, which represent con icts, can exist between the same type of players. In other words, my own type may be my friend or enemy, but the other typee is always my enemy in a positive assortative matching network. As mentioned in Remark 1, all Nash equilibrium network con gurations consists of cliques. Therefore, discussing Nash equilibrium regarding positive assortative matching, I consider only the network con gurations, which consist of cliques made of the same type of players.

Remark 4 Every Nash equilibrium positive assortative matching consists of cliques $C_1; C_2; ::::; C_m$, , and there are only one type of players in each clique.

Once again, by de nition of clique, g_{ij} =1 if $i; j$ 2 C_k and g_{ij} =-1 if i 2 C_k; j 2 C_l where $k \neq l$. Among positive assortative matching, there are special cases where a type of players dominating the other type of players. For example, suppose there are three alliances.

(a) (b) (c)

(d) (e)

Figure 5: Strong dominant positive assortative matching

Two alliances consist of incumbents, and one alliance consists of challengers. Among the incumbents' alliances, one has the strongest marketing power, and the other has the second strongest marketing power. The challenger's alliance is the weakest one, so they always lose their customers to the other alliances.

Let us see strong dominant positive assortative matching first. As described in Figure 5, there are five cases of Nash equilibrium strong dominant positive assortative matching in the four players model. These five configurations can be Nash equilibria according to the parameters" + and a. The conditions are specified in Appendix.

For complete strong dominant positive assortative matching (CSPM) to be a Nash equilibrium, every deviation has to give lower payoff for each payer than this strategy profile. Similar to the Utopia strategy, there exist particular deviations from SPM always more profitable than the other deviations. This particular deviations vary from case to case, so we sorted these in Lemma 2 below. Note that any deviation strategy by a weak player is always not profitable in every SPM. Lemma 3 proves it in Appendix. Simply speaking, in any SPM, if the weak player extends negative links to the strong player, nothing is changed, but only the cost " happens. If this player extends negative links to the other weak players, it is also not profitable. The payoff from the conflicts is always zero or negative because $f((n_w \quad x)\ _w; (n_w \quad 1)\ _w)$ 0 when x 1. Similarly, regarding WPM, any strong players do not have a profitable deviation.

Lemma 2 Complete strong dominant possitive assortative matching g is a N.E. if and only if

(i) when $(n_s \quad 1) \quad_s \quad n_w \quad_w$, $u_i(g) \quad u_i(g + g^+_{ij})$, or

(ii) when $(n_s \quad 1) \quad_s < n_w \quad_w$, $u_i(g) \quad u(g + g^+_{ij})$ and $u_i(g) \quad u_i(g + \sum_{k2K} g_{ik} + \sum_{l2N_w} g^+_{il})$

for all $i \; 2 \; N_s$, $j \; 2 \; N_w$, $K \quad N_s$.

$u_i(g) \quad u_i(g + \sum_{j2J} g_{ij})$ for all $J \quad N_w$ is the condition for each strong player not to embrace some of the weak players. If $u_i(g) \quad u_i(g + g_{ij})$, then $u_i(g) \quad u_i(g + \sum_{j2J} g_{ij})$ for all $J \quad N_w$. It is because of the property $\frac{@^2 f}{@ \; ^2} < 0$ decreasing marginal extraction to network strength. Note that every contest success functions satisfy the property. The more detailed proof is in the appendix. Stories about the deviations are as follows. When one group exploits another group, the attacking group seeks turncoats. The newly formed positive relationship with the turncoats is for exploiting the rest of the exploited group members. On the other hand, $g \quad \sum_{k2K} g_{ik} + \sum_{j2N_w} g^+_{ij}$ means to initiate new con icts against the same kind of players by taking advantage of the other group. Thus, player i is a casting voter here. If player i betrays the old strong type friends, then the new coalition of and the new weak type friends gets a higher network strength than the old friends' network strength. This condition also means that this deviation is pro table only when this betrayal e ciently weakens the remaining strong players' network strength.

In the rst case (i), any deviation switching alliance is an unattractive choice as mentioned in Lemma 5. The new weak friend is not useful to increase the network strength. Also, the new enemy who was an old friend before the deviation is still stronger than the new friend. On the other hand, in the second case (ii), the old friends are weaker than the new friends after the deviation. However, these new friends are not as helpful as the old friends to increase the aggregate strength. It means there is a tradeo in switching alliances.

From Lemma 2, it is possible to derive the necessary and su cient condition concerning the parameters for CSPM to be a Nash equilibrium, because CSPM is a special case of SPM. It is also a su cient condition that there exists a Nash equilibirum SPM. Also, there is a condition that all Nash equilibria are SPM.

Theorem 2 For $" + \quad < f \quad (n_s; n_s \quad 1)$, there exists $a^a(n_s; n_w; " + \quad)$ such that all Nash equilibria are strong dominant positive assortative matching if $a \quad a^a$, and for $" + \quad < \; f$ there also exists $a^e(n_s; n_w; " + \quad)$ such that CSPM is a Nash equilibrium if and only if $a \quad a^e$.

Proposition 2 indicates that positive assortative matching can be a Nash equilibrium only if there is a signi cant di erence between $_s$ and $_w$. Furthermore, if the di erence gets larger, every Nash equilibrium con guration is strong dominant positive assortative matching. The result implies that rst, gathering and discrimination by the strong player can happen only when the strong people are competitive enough compared to the discriminated people. Second, in addition to this situation where strong players are quite stronger than weak players, if the con ict cost is small enough, only strong dominant positive assortative matchings can be Nash equilibria.

It is also available to consider extreme cases such as $\to 1$ and $" + < \frac{1}{2}$.

Corollary 6 When $f = h$ and $\to 1$, complete strong dominant positive assortative matching is a Nash equilibrium for a such that $n_s a > n_w$ and $" + < \frac{1}{2}$.

Remark 5 When $f = h$, suppose $\to 0$. For any $" + > 0$, any strong dominant positive assortative matching is not a Nash equilibrium regardless of a.

High ampli es the e ectiveness of extraction from the di erence between $_s$ and $_w$. Therefore, su ciently large allows any a to satisfy the condition for a Nash equilibrium. Remark 5 is derived from Corollary 5 that when $\to 0$, Utopia network is the unique Nash equilibrium. When is su ciently low, complete strong dominant positive assortative matching cannot be a Nash equilibrium because the Utopia network is the unique Nash equilibrium in this case as shown in Corollary 5. The relative simulations are in Appendix.

To glance a case where is moderate, let's consider the special case that $= 1$ and $" + \to 0$.

Corollary 7 When $f = h$, suppose $= 1$ and $" +$ is small enough. The complete strong dominant positive assortative matching is a N.E. if and only if

i) When $n_s = 2$ and $n_w = 2$, $a > a^e(2; 2; " + \to 0) = 1{:}5$, or

ii) Otherwise, $a > a^e(n_s; n_w; " + \to 0) = \frac{1}{2n_s}(\; 1 + \sqrt{12n_w^2 \; 4n_w + 1})$.

Corollary 7 is consistent with Lemma 2. The rst condition, when $n_s = 2$ and $n_w = 2$, is derived from the condition $u_i(g) \; u_i(g + g_{ij}^+)$, and the other condition is derived from $u_i(g) \; u_i(g + \sum_{k \in K} g_{ik} + \sum_{j \in N_w} g_{ij}^+)$. If n_s increases then a^e decreases, and if n_w increases then a^e increases.

Before going to the next part regarding weak dominant positive assortative matching, let's mention a condition where any disassortative matching is not in equilibrium. Theorem 2 states the condition where all Nash equilibriua are strong dominant positive assortative matchings. It means that any other network con gurations, including disassortative matching, cannot be a Nash equilibrium when the condition for Theorem 2 is met. From this point, it is possible to derive a condition for disassortative matching not to be in equilibrium.

Remark 6 If a is large enough, any disassortative matching is not in equilibrium.

Figure 6: Weak dominant positive Assortative Matching

Now let's consider weak dominant positive assortative matching. In the example with four players, there is only one weak dominant positive assortative matching configuration as drawn in Figure 6. This undirected network is the same to Figure 5 (c), but the strategy profile denoted by the directed network is different. In Figure 5 (c), the strong player extends directed negative links, but in Figure 6, the weak players extend directed negative links. For each negative link to be profitable, the players who extend the negative links should have larger aggregate strength. Therefore, in Figure 5 (c), the strong player should have his intrinsic power larger than 3, and in Figure 6, the strong player should have his intrinsic power less than 3. For the configuration in Figure 6 to be a Nash equilibrium, the weak players should not have any incentive to deviate from the weak dominant positive assortative matching strategy. We specify the condition in Appendix.

Like the strong dominant positive assortative matching, there is a condition for the weak dominant positive assortative matching to be a Nash equilibrium for an arbitrary number of players. Weak dominant positive assortative matchings are Nash equilibria when the difference between the types is not significant. $u_i(g) \geq u_i(g + g^+_{ij})$ and $u_i(g) \geq u_i(g + \sum_{k \in K} g^-_{ik} + \sum_{j \in N_s} g^+_{ij})$ In particular, Proposition 2 focuses on the complete weak dominant positive assortative matching. It is about the condition for complete weak dominant positive assortative matching to be a Nash equilibrium.

Proposition 2 For any n_w and n_s such that $n_w > 2n_s - 1$, and for small enough $\epsilon' + \epsilon > 0$, there exists $\underline{a}(n_s; n_w)$ such that the complete weak dominant positive assortative matching is a Nash equilibrium if and only if $a \geq \underline{a}$.

Weak dominant positive assortative matchings describe situations in which the weak players form a majority in society because of their number. Proposition 2 implies that this network configuration can be a Nash equilibrium only when the difference between σ_s and σ_w is not so large. If it is significantly large, then there is always a weak player who has an incentive to form new friendships with the strong type enemies. Note that weak dominant positive assortative matchings are not all Nash equilibria even when the difference between the types is small enough. When the difference is small, other Nash equilibrium configurations also exist, such as Utopia, disassortative matching, and even strong dominant positive assortative matching. When a is small, it brings a similar result to the homogeneous player model. On the other hand, the result is not opposed to the result when a is large.

In Corollary 8 and Remark 7, we show how the extreme assumption such as $\to 1$ or $\to 0$ changes the condition of the Nash equilibrium weak dominant positive assortative matching.

Corollary 8 When $f = h$, suppose $\to 1$. Then complete weak dominant positive assortative matching cannot be a Nash equilibrium for any a.

Remark 7 When $f = h$, suppose $\to 0$. Then any weak dominant positive assortative matching cannot be a Nash equilibrium for any a.

According to Corollary 8, when the extraction technology is highly efficient, the strong players are never discriminated. For the weak type, it is always beneficial to embrace the talented minority whose intrinsic power is $_s$ and to establish other conflicts with some of the old friends. If the extraction technology is highly efficient, the weak players always prefer to betray the old friends with help from the strong players. Similar to Remark 5, Remark 7 is derived from Corollary 5 that when $\to 0$, Utopia network is the unique Nash equilibrium. If the technology is extremly not efficient, each weak player give up the extraction, and choose peace in the Utopia network.

4 Conclusion

The paper has built upon Hiller (2017) by studying different equilibrium configurations. First, we investigate all Nash equilibrium network configurations in the example with four players. While there are only two network configurations with four homogeneous players that can be sustained in equilibrium, there are much more network configurations sustained in equilibrium with two types of players. In the model with four heterogeneous players, there are thirteen network configuration categories in equilibrium. Except for the network configurations, there is no other network configuration in equilibrium in this four players model.

After we identify these thirteen Nash equilibrium configurations, we sorted them to three categories: Utopia network, positive assortative matching, and disassortative matching. In the case of an arbitrary number of players and two types, we derive the generalized condition for existence of an equilibrium exhibiting the Utopia networks and the positive assortative matching. In both homogeneous and heterogeneous model, the Utopia network can be a Nash equilibrium or the unique Nash equilibrium if the con ict cost is signi cant. If the di erence between intrinsic strengths gets larger, then the level of con ict cost for the Utopia network to be a Nash equilibrium also increases. Generally, when the heterogeneity is introduced, strong players have an incentive to deviate from the Utopia networks. However, if the number of strong players increases, then the Utopia network can be sustainable in the lower level of con ict cost. It is because when there are lots of strong players, each of them holds each other in check. It is not pro table for the strong players to extend a negative link to a weak player when the weak player has lots of strong friends. Therefore, while introducing strong players into a network can impede cooperation in the begging, but adding more strong players also promotes cooperation.

In the boundary of positive assortative matching, there is strong dominant positive assortative matching, weak dominant positive assortative matching, and the other positive assortative matching, which are neither strong dominant, not weak dominant. If the strong player's intrinsic strength is su ciently larger than the weak player's intrinsic strength, all Nash equilibrium network con gurations are these positive assortative matching. We also show that if the di erence between the intrinsic strengths is small enough, then complete weak dominant positive assortative matching is a Nash equilibrium. Naturally, disassortative matching cannot be a Nash equilibrium when the di erence is large enough.

References

Abowd, John M, Francis Kramarz, Paul Lengermann, and Sebastien Perez-Duarte. 2004. \Are good workers employed by good rms? A test of a simple assortative matching model for France and the United States." Unpublished Manuscript.

Andrews, Martyn J, Len Gill, Thorsten Schank, and Richard Upward. 2006. \High wage workers and low wage rms: negative assortative matching or statistical artefact?" Diskussionspapiere.

Andrews, Martyn J, Leonard Gill, Thorsten Schank, and Richard Upward. 2012. \High wage workers match with high wage rms: Clear evidence of the e ects of limited mobility bias." *Economics Letters*, 117(3): 824{827.

Antal, Tibor, Paul L Krapivsky, and Sidney Redner. 2006. \Social balance on networks: The dynamics of friendship and enmity." Physica D: Nonlinear Phenomena, 224(1-2): 130{136.

Becker, Gary S. 1973. \A theory of marriage: Part I." Journal of Political economy, 81(4): 813{846.

Bozbay, Irem, and Alberto Vesperoni. 2018. \A contest success function for networks." Journal of Economic Behavior & Organization, 150: 404{422.

Brown, David JJ. 1979. \The structuring of Polopa feasting and warfare." Man, 712{733.

Calvo-Armengol, Antoni, and Rahmi Ilk lc. 2009. \Pairwise-stability and Nash equilibria in network formation." International Journal of Game Theory, 38(1): 51{79.

Cartwright, Dorwin, and Frank Harary. 1956. \Structural balance: a generalization of Heider's theory." Psychological review, 63(5): 277.

Celikaksoy, Aycan, Helena Skyt Nielsen, and Mette Verner. 2006. \Marriage migration: just another case of positive assortative matching?" Review of Economics of the Household, 4(3): 253{275.

Chiappori, Pierre-Andre, Sonia Ore ce, and Climent Quintana-Domeque. 2012. \Fatter attraction: anthropometric and socioeconomic matching on the marriage market." Journal of Political Economy, 120(4): 659{695.

Cisneros-Velarde, Pedro, and Francesco Bullo. 2019. \Signed Network Formation Games and Clustering Balance." Dynamic Games and Applications, 1{15.

Davis, James A. 1967. \Clustering and structural balance in graphs." Human relations, 20(2): 181{187.

Ghatak, Maitreesh. 1999. \Group lending, local information and peer selection." Journal of development Economics, 60(1): 27{50.

Ghatak, Maitreesh. 2000. \Screening by the company you keep: Joint liability lending and the peer selection e ect." The Economic Journal, 110(465): 601{631.

Goyal, Sanjeev, Hoda Heidari, and Michael Kearns. 2014. \Competitive contagion in networks." Games and Economic Behavior.

Grandjean, Gilles, Daniela Tellone, and Wouter Vergote. 2017. \Endogenous network formation in a Tullock contest." Mathematical Social Sciences, 85: 1{10.

Greenwood, Jeremy, Nezih Guner, Georgi Kocharkov, and Cezar Santos. 2014. “Marry your like: Assortative mating and income inequality.” American Economic Review, 104(5): 348–53.

Harary, Frank, et al. 1953. “On the notion of balance of a signed graph.” The Michigan Mathematical Journal, 2(2): 143–146.

Harary, Frank, et al. 1955. “On local balance and N-balance in signed graphs.” The Michigan Mathematical Journal, 3(1): 37–41.

Heider, Fritz. 1946. “Attitudes and cognitive organization.” The Journal of psychology, 21(1): 107–112.

Hiller, Timo. 2017. “Friends and enemies: A model of signed network formation.” Theoretical Economics, 12(3): 1057–1087.

Hirshleifer, Jack. 1989. “Conflict and rent-seeking success functions: Ratio vs. difference models of relative success.” Public choice, 63(2): 101–112.

Jackson, Matthew O, and Asher Wolinsky. 1996. “A strategic model of social and economic networks.” Journal of economic theory, 71(1): 44–74.

Jackson, Matthew O, and Stephen Nei. 2015. “Networks of military alliances, wars, and international trade.” Proceedings of the National Academy of Sciences, 112(50): 15277–15284.

Johnson, Terence R. 2013. “Matching through position auctions.” Journal of Economic Theory, 148(4): 1700–1713.

König, Michael D, Dominic Rohner, Mathias Thoenig, and Fabrizio Zilibotti. 2017. “Networks in conflict: Theory and evidence from the great war of africa.” Econometrica, 85(4): 1093–1132.

Kremer, Michael. 1993. “The O-ring theory of economic development.” The Quarterly Journal of Economics, 108(3): 551–575.

Laumann, Edward O, and Franz Urban Pappi. 1973. “New directions in the study of community elites.” American Sociological Review, 212–230.

Legros, Patrick, and Andrew F Newman. 2007. “Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities.” Econometrica, 75(4): 1073–1102.

Li, Sanxi, Hailin Sun, and Pu Chen. 2013. “Assortative matching of risk-averse agents with endogenous risk.” *Journal of Economics*, 109(1): 27–40.

Luce, R Duncan. 1950. “Connectivity and generalized cliques in sociometric group structure.” *Psychometrika*, 15(2): 169–190.

Mendes, Rute, Gerard J Van Den Berg, and Maarten Lindeboom. 2010. “An empirical assessment of assortative matching in the labor market.” *Labour Economics*, 17(6): 919–929.

Roy, Andrew Donald. 1951. “Some thoughts on the distribution of earnings.” *Oxford economic papers*, 3(2): 135–146.

Shimer, Robert, and Lones Smith. 2000. “Assortative matching and search.” *Econometrica*, 68(2): 343–369.

Siow, Aloysius. 2015. “Testing Becker's theory of positive assortative matching.” *Journal of Labor Economics*, 33(2): 409–441.

Tullock, Gordon. 1967. “The welfare costs of tariffs, monopolies, and theft.” *Economic Inquiry*, 5(3): 224–232.

Tullock, Gordon. 1980. “Efficient Rent Seeking. JM Buchanan, RD Tollison, G. Tullock, eds., Toward a Theory of the Rent Seeking Society.” A & M University Press, College Station, TX: Texas.

Van Tassel, Eric. 1999. “Group lending under asymmetric information.” *Journal of Development Economics*, 60(1): 3–25.

Appendix A. Proof

The properties of $h(y_i; y_j; \ )$

The normalized contest success function $h(y_i; y_j; \ )$ satisfies the properties of $f(y_i; y_j)$ as follows.

1. $\frac{df}{dy_i} > 0$: f increases in y_i.

$$\frac{dh}{dy_i} = \frac{y_i^{\ \ 1} y_j}{(y_i + y_j)^2} > 0$$

2. $\frac{df}{dy_j} < 0$: f decreases in y_j.

$\frac{dh}{dy_j} = \frac{y_i\, y_j \quad 1}{(y_i + y_j)^2} < 0$

3. f is concave on the range of y_i.

$\frac{d^2 h(y_i; y_j)}{dy_i^2} = \frac{y_i \quad {}^2 y_j\, ((\quad +1)\, y_i \quad (\quad 1) y_j)}{(y_i + y_j)^3} < 0$ when $y_i > y_j$.

4. f is homogeneous of degree 0.

$$h(\ y_i;\ y_j;\ \ ) = \frac{(\ y_i)}{(\ y_i) \ + (\ y_j)} \quad \tfrac{1}{2} = \frac{y_i}{y_i + y_j} \quad \tfrac{1}{2} = h(y_i; y_j;\ \ )$$

In addition, in $h(y_i; y_j;\ )$, represents the efficiency in technology of extraction. the larger is, the bigger the size of extraction is for the same y_i and y_j for $y_i > y_j$.

$\frac{dh}{d} = \frac{y_i\, y_j\, (\log(y_i) \quad \log(y_j))}{(y_i + y_j)^2} > 0$ when $y_i > y_j$.

Remark 3

(1) Suppose $\varepsilon + \ > f\ (n \quad 1; 1)$. Let a player i have a negative link, i.e., $g_{ij} = \quad 1$. Now the benefit from having the negative link is $f\,(y_i(g); y_j(g))$: For g with n homogeneous players, $f\,(n \quad 1; 1)$ is the maximum extraction value. The cost of this negative link is given by $\varepsilon + \ $.

Let player i deviate from the negative link to the positive link, i.e., $g_{ij} = 1$: This deviation is profitable for player i. First, directly, $\varepsilon + \ > f\ (n \quad 1; 1) \quad f\,(y_i(g); y_j(g))\ \forall y_i; y_j$. Also, indirectly, this deviation increases the extractions generated on the other negative links. The positive link increases player i's network strength $y_i(g)$ to $y_i(g') = y_i(g)+1$. Then, on another negative link with player l, the amount of extraction increases because $f\,(y_i(g)+1; y_l(g)) > f\,(y_i(g); y_l(g))$. Therefore, the deviation from $g_{ij} = \quad 1$ to $g_{ij} = 1$ is always profitable given the condition. Thus there is no conflict when $\varepsilon + \ > f\ (n \quad 1; 1)$.

(2) Suppose $\varepsilon + \quad f\,(n \quad 1; 1)$. Let a player i's strategy g_i be given by $g_i = \{ g_{i;1}; g_{i;2}; ::::; g_{i;i \quad 1}; g_{i;i+1}; ::::; g_{i;n}\} = \{1; ::::; 1\}$:

Let player i deviate to $g_i + \sum_{j \in J} g_{ij}$ where $J \in N \setminus \{i\}$: Take $|J| = 1$, then $y_i = n \quad 1$ and $y_j = n \quad 1$ where $j \in J \quad N$: In this case, $f\,(y_i; y_j) = 0$. But $\varepsilon + \ > \ 0$. Thus, it is not profitable to deviate to the negative link.

Let's take $|J| \quad 2$, then $y_i \quad n \quad 2$ and $y_j = n \quad 1$, so $f\,(y_i; y_j) < 0$ and $\varepsilon + \ > \ 0$. Thus, it is not profitable to deviate to the negative link, hence Utopia network is a Nash equilibrium.

Lemma 3 In any network formation strategy g, which has a clustering, $u_i(g) \quad u_i(g + \sum_{k \in K} g_{ik})$ where $K \quad N_i^+$ and $\forall \ _k = \ _i$.

Proof. This lemma means that initiating new conflicts with the same type of players in the same clique is always unprofitable. To prove it, consider the deviation strategy $g + \sum_{k \in K} g_{ik}$. Before the deviation, in the network configuration g, a player i has $y_i(g)$, and a player k in the set K also has $y_i(g)$. After the deviation, $y_i(g + \sum_{k \in K} g_{ik}) = y_i \quad |K| \quad _i$. However,

$y_k(g + \sum_{k \in K} g_{ik}) = y_i \quad _i$. $y_i \quad |K| \quad _i$ is the same to or less than $y_i \quad _i$. Hence, $f(y_i \quad |K| \quad _i; y_i \quad _i) \quad 0$. ■

Lemma 4 Suppose a network configuration with a strategy g consists of two cliques,[12] and the players in each clique has the same instrinsic strength. If $u_i(g) < u_i(g + \sum_{j \in J} g^+_{ij})$ for some set J such that $J \quad N_i$ and $|J| \quad 2$, then $u_i(g) < u_i(g + g^+_{ij})$ where $j \in N_i$.

Proof. Let C_1 and C_2 denote each clique. The players in C_1 have an identical intrinsic power $_1$, and those in C_2 have $_2$. Then player's aggregate power is $|C_1| \quad _1$ in C_1, and $|C_2| \quad _2$ in C_2. Suppose $|C_1| \quad _1 > |C_2| \quad _2$ without loss of generality. By Remark 2, a corresponding strategy profile g to this network configuration is as follows. If player i is in C_1, $g_{i;j} = 1$ for $j \in C_1$, and $g_{;j} = \quad 1$ for $j \in C_2$. If player i is in C_2, $g_{i;j} = 1$ for $j \in N \setminus \{i\}$.

Note that this lemma is only about the players in C_1. For player i in C_2, $u_i(g) = u_i(g + \sum_{j \in J} g^+_{i;j})$; because $g_{ij} = 1$ for $j \in J$ so $g + \sum_{j \in J} g^+_{i;j} = g$.

Let's consider a player i in $|C_1|$. Her utility regarding the deviation $g + \sum_{j \in J} g^+_{ij}$ is as follows.

$$u_i(g + \sum_{j \in J} g^+_{ij}) = (|C_2| \quad |J|) f(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2) \quad (|C_2| \quad |J|)(\varepsilon + \quad)$$

The first and second derivative of $u_i(g + \sum_{j \in J} g^+_{i;j})$ with respect to $|J|$ are as follows.

$$\frac{\partial \mu}{\partial |J|} = \quad f(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2) + \quad _2(|C_2| \quad |J|) f'(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2) + \varepsilon +$$

$$\frac{\partial^2 u_i}{\partial |J|^2} = \quad 2 \quad _2 f'(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2) + \quad ^2_2(|C_2| + |J|) f''(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2)$$

The second derivative is less than zero when $f''(|C_1| \quad _1 + |J| \quad _2; |C_2| \quad _2) < 0$. By assumption, $\frac{\partial^2 f}{\partial y^2} < 0$ so the second derivative $\frac{\partial^2 u_i}{\partial |J|^2}$ is always less than zero. If the second derivative is less than zero, it is impossible that $u_i(g + g^+_{i;j}) < u_i(g) < u_i(g + \sum_{j \in J} g^+_{i;j}$ for $|J| \quad 2$. Therefore, if $u_i(g + g^+_{i;j}) < u_i(g)$, then $u_i(g + \sum_{j \in J} g^+_{i;j} < u_i(g)$ for $|J| \quad 2$.

■

Lemma 4 implies that it is enough check only the case of $g + g^+_{i;j}$ instead of checking every case of $|J| \quad 2$ to check an incentive for player i in C_1 to deviate from the strategy g.

Lemma 5 Suppose there is a network configuration consisting of two cliques C_1 and C_2. The players in C_1 have the same $_1$, and those in C_2 have $_2$, such that $_1 \quad _2$. If $(|C_1| \quad 1) \quad _1 \quad |C_2| \quad _2$, For any $g'' = g + \sum_{j \in J_2} g^+_{ij} + \sum_{k \in K_2} g_{ik}$, there exists $g' = g + \sum_{j \in J_1} g^+_{ij}$ or

[12] It implies that g is structurally balanced.

$g' = g + \sum_{k\in K_1} g_{ik}^-$, such that $u_i(g') \le u_i(g'')$, where $J_1 \& J_2 \subset C_2 = N_i^-$ and $K_1 \& K_2 \subset C_1 \cap \{i\} = N_i^+$.

Proof.

The corresponding strategy to this network configuration is as mentioned in the proof of Lemma 4. Player i's network strength is given by $y_i(g'') = (|C_1| - |K_2|)\alpha_1 + |J_2|\alpha_2$. Now, let's consider another network:

$$\tilde{g} = g'' + g_{ij}^- + g_{ik}^+$$

where $j \in J_2$ and $k \in K_2$. Then

$$\begin{aligned} y_i(\tilde{g}) &= (|C_1| - |K_2| + 1)\alpha_1 + (|J_2| - 1)\alpha_2 \\ &= (|C_1| - |K_2|)\alpha_1 + |J_2|\alpha_2 + \alpha_1 - \alpha_2 \\ &\ge y_i(g'') \end{aligned}$$

In g'', for $j \in J_2 \setminus N_i^-(\tilde{g})$, $k \in K_2 \setminus N_i^+(\tilde{g})$ player j's network strength is smaller than player k's network strength as $y_j(g'') = y_j(\tilde{g}) = |C_2|\alpha_2 \le n_k(g'') = n_k(\tilde{g}) = (|C_1| - 1)\alpha_1$. The number of the negative links extended by player i is the same in both g'' and $\tilde{g}$. Therefore,

$$\begin{aligned} u_i(\tilde{g}) = &(|C_2| - |J_2|) f\left((|C_1| - |K_2| + 1)\alpha_1 + (|J_2| - 1)\alpha_2; |C_2|\alpha_2\right) \\ &+ |K_2| f\left((|C_1| - |K_2| + 1)\alpha_1 + (|J_2| - 1)\alpha_2; (|C_1| - 1)\alpha_1\right) \\ &+ f\left((|C_1| - |K_2| + 1)\alpha_1 + (|J_2| - 1)\alpha_2; |C_2|\alpha_2\right) \\ &- f\left((|C_1| - |K_2| + 1)\alpha_1 + (|J_2| - 1)\alpha_2; (|C_1| - 1)\alpha_1\right) \\ &- (|C_2| - |J_2| + |K_2|)(\varepsilon + \gamma) \\ u_i(g'') = &(|C_2| - |J_2|) f\left((|C_1| - |K_2|)\alpha_1 + |J_2|\alpha_2; |C_2|\alpha_2\right) \\ &+ |K_2| f\left((|C_1| - |K_2|)\alpha_1 + |J_2|\alpha_2; (|C_1| - 1)\alpha_1\right) \\ &- (|C_2| - |J_2| + |K_2|)(\varepsilon + \gamma) \end{aligned}$$

Therefore, $\tilde{g}$ gives a utility higher than or equal to g'. $u_i(\tilde{g}) = u_i(g'')$ can happen when every player is homogeneous. It is possible to iterate this procedure until $\tilde{g} = g + \sum_{j\in J_1} g_{ij}^+$ or $\tilde{g} = g + \sum_{k\in K_1} g_{ik}^-$. ■

Lemma 5 implies that it is enough to calculate whether $u_i(g) > u_i(\tilde{g})$ or not, to check whether player i has an incentive to deviate from g.

Lemma 1

We will check whether each deviation gives higher utility than the suggested strategy profile for each player. As a result of this procedure, it is possible to find a condition which makes any deviations unprofitable.

Once again, in the network g in equilibrium, there are n_1 players in C_1 who extend negative links to n_2 players in C_2 where $n_1 > n_2 = n - n_1$. Except for these negative links, there are only positive links. The full strategy profile of g is as suggesteed in the proof of Lemma 4.

First, there are three kinds of deviation for any player i in C_1. (i) Player i can extend positive links to players in C_2 and negative links to players in C_1 together. (ii) She can only extend negative links in C_1. Lastly, (iii) she can only extend positive links in C_2.

When the first type of deviation (i) can be profitable than g, then the second type (ii) or the third type deviation (iii) is also profitable than g by Lemma 5. In this homogeneous model with two cliques, Lemma 5 is applicable as the assumptions are satisfied. Therefore, to check whether g is a Nash equilibrium or not, it is enough not to check the first type deviation (i). Second,by Lemma 3, the second type of deviations (ii) are always non-profitable. Third, regarding (iii), it is enough to check only whether $g + g^{+}_{ij}$, where $j \in C_2$, gives a higher utility than g following Lemma 4

$n_2 f(n_1; n_2) - n_2(\epsilon + \ ) \geq (n_2 - 1) f(n_1 + 1; n_2) - (n_2 - 1)(\epsilon + \ )$ is a different expression of $u_i(g) - u_i(g + g^{+}_{ij})$ using the parameters given in the model. Therefore, if and only if this condition is satisfied, then g is a Nash equilibrium.

Second, the players in C_2 can only change some of their positive links to the negative links in any deviation strategy. However, by Lemma 3, extending the negative links to other players in C_2 is not profitable. Also, extending the negative links to players in C_1 does not change the undirected network g, but generates additional conflict costs. Therefore, for the players in C_2, g describes the best response strategy.

Proposition 1

Let's use the condition $n_2 f(n_1; n_2) - (n_2 - 1) f(n_1 + 1; n_2) \geq \epsilon + \ $ in Lemma 1. This condition is equivalent to

$$f(n_1; n_2) - (n_2 - 1)(f(n_1 + 1; n_2) - f(n_1; n_2)) + \epsilon +$$

Given n_2 and $" + \ < f\ (n \ 1;1)$, if n_1 is large enough, then this equation holds.

Also, we will show that there always exists $n_1(n_2;" + \ ) \ n$. Let's consider the case where $n_1 = n \ 1$. In this case, this con guration is always a Nash equilibrium.[13] A player i in C_1 does not have an incentive to deviate from this strategy because when $f(n \ 1;1) > " + \ $, $g + g^+_{ij}$ where $j \ 2 \ C_2$ is unpro table. Also, $g + \sum_{j 2 J} g_{ij}$ where $J \ C_1 n f i g$ is always not pro table by Lemma 3. Lastly, $g + g^+_{ij} + \sum_{j 2 C_1 n f i g} g_{ij}$ where $J \ C_1 n f i g$ and $k \ 2 \ C_2$ is also not pro table than g too in the same way.

The player in C_2 also do not have an incentive to change his strtegy by Remark 2.

Theorem 1

(a) If and only if condition when Utopia network is a Nash equilibrium: By de nition, g is a Nash equilibrium if and only if there is no $g^0_i \ 2 \ G_i$ such that $u_i(g^0_i; g_{\ i}) > u_i(g)$ for all i. In Utopia network, a player i's strategy $g_i = f g_{i;1}; g_{i;2}; ::::; g_{i;n} g = f 1; ::::; 1 g$:

Any weak player does not have an incentive to deviate from the Utopia network strategy. First, $g^{00} = \ g + g_{ij}$ where $j \ 2 \ N_w$ is unpro table. If weak player i betrays one strong player j by extending one negative link, it is an unpro table deviation because $y_i(g^0) = (n_s \ 1)\ _s + n_w\ _w < y_j(g^0) = n_s\ _s + (n_w \ 1)\ _w$, so $f(y_i(g^0); y_j(g^0)) < 0$. If weak player i betrays one weak player j by extending one negative link, it is also unpro table by Lemma 3. If the weak player i betrays more than one player, then it is always unpro table, too. When she betrays x_s strong players and x_w weak players, $y_i(g^0) = (n_s \ x_s)\ _s + (n_w \ x_w)\ _w$. However, any other player j's network strength is $y_j(g^0) = n_s\ _s + (n_w \ 1)\ _w$, so $f(y_i(g^0); y_j(g^0)) < 0$. Therefore, any weak players do not have an incentive to deviate from g.

On the other hand, each strong player does not have an incentive to deviate from Utopia network strategy if and only if $u(g) \ u(g^{00}) = \ g + g_{ij}$. It is possible to transform $u(g) \ u(g^{00})$ to $" + \ \ f(n_s\ _s + (n_w \ 1)\ _w; (n_s \ 1)\ _s + n_w\ _w)$. First, a player i's deviation $g_i + \sum_{j 2 J} g_{ij}$ where $J \ 2 \ N_s n f i g$ is always unpro table by Lemma 3. Second, let's consider two other deviations: (i) $g = \ g + \sum_{j 2 J} g_{ij}$ for $J \ N_s [\ N_w$ and $J \setminus N_s \ 6= \ ?$ and $J \setminus N_w \ 6= \ ?$, and (ii) $g^0 = \ g + \sum_{k 2 J \setminus N_s} g^+_{ik}$. Note that g^0 always gives higher utility than g. The reason is as follows. Once again, x_s is the number of the betrayed strong players and x_w is the number of the betrayed weak players. In g, for $j \ 2 \ N_s$, $y_i(g) = (n_s \ x_s)\ _s + (n_w \ x_w)\ _w$, and $y_j(g) = (n_s \ 1)\ _s + n_w\ _w$. Then, $f(y_i(g); y_j(g)) < 0$. Therefore, for the strong player i, g^0 always gives more utility than g. Lastly, let's consider the deviation $g^0 = \ g + \sum_{l 2 L} g_{il}$ where

[13] This is what Hiller (2017) proved in his Proposition 2

$L \in N_w$. Note that $g' = g + \sum_{l \in L} g_{il} = g + \sum_{k \in J \cap N_s}^{+} g_{ik}$. The utility from g' is as follows.

$$
\begin{aligned}
u_i(g') &= \sum_{l \in L} f(y_i(g'); y_l(g')) - \varepsilon \\
&= x_w \, f(n_s{}_s + (n_w - x_w){}_w; (n_s - 1){}_s + n_w{}_w) - \varepsilon
\end{aligned}
$$

By using the method used in Lemma 4,

$$
\begin{aligned}
\frac{\partial u(g')}{\partial x_w} &= f(n_s{}_s + (n_w - x_w){}_w; (n_s - 1){}_s + n_w{}_w) - \varepsilon \\
&\quad - x_w{}_w f'(n_s{}_s + (n_w - x_w){}_w; (n_s - 1){}_s + n_w{}_w) \\
\frac{\partial^2 u(g')}{\partial x_w^2} &= -2{}_w f'(n_s{}_s + (n_w - x_w){}_w; (n_s - 1){}_s + n_w{}_w) \\
&\quad + {}_w^2 f''(n_s{}_s + (n_w - x_w){}_w; (n_s - 1){}_s + n_w{}_w):
\end{aligned}
$$

When $f'' < 0$; $\frac{\partial^2 u(g')}{\partial x_w^2} < 0$. Therefore, if $u_i(g) \;\; u_i(g')$ for $x_w = 1$, then $u_i(g) \;\; u_i(g')$ for $x_w \;\; 2$.

(b) If and only if condition when Utopia network is the unique Nash equilibrium: In the network with heterogeneous players given the parameters $n_s; n_w; {}_s$ and ${}_w$, $f(n_s{}_s + (n_w - 1){}_w; {}_w)$ is the maximum of extraction. This is because $n_s{}_s + (n_w - 1){}_w$ is the maximum network strength and ${}_w$ is the minimum network strength given n_s, n_w, ${}_s$, and ${}_w$. First, if $\varepsilon + \; > f(n_s{}_s + (n_w - 1){}_w; {}_w)$, then any networks including negative links cannot be Nash equilibria. In these networks, players who are extending negative links have an incentive to change these links to the positive links. However, Utopia network is still a Nash equilibrium because $\varepsilon + \; > f(n_s{}_s + (n_w - 1){}_w; {}_w) > f(n_s{}_s + (n_w - 1){}_w; (n_s - 1){}_s + n_w{}_w)$. Thus, Utopia network is the unique Nash equilibrium.

Second, if Utopia network is the unique Nash equilibrium, then $\varepsilon + \; > f(n_s{}_s + (n_w - 1){}_w; {}_w)$. As shown above, Utopia network is a Nash equilibrium iff $\varepsilon + \;\; f(n_s{}_s + (n_w - 1){}_w; (n_s - 1){}_s + n_w{}_w)$. If $\varepsilon + \; < f(n_s{}_s + (n_w - 1){}_w; (n_s - 1){}_s + n_w{}_w)$, then the Utopia network is not a Nash equilibrium. If $\varepsilon + \;\; f(n_s{}_s + (n_w - 1){}_w; {}_w)$, then there is another Nash equilibrium network configuration where $n - 1$ players are in a clique C_1 and they extend the negative links to the other player in C_2. If $\varepsilon + \; > f(n_s{}_s + (n_w - 1){}_w; {}_w)$, then any networks containing negative links cannot be a Nash equilibrium. Therefore, if the Utopia network is the unique Nash equilibrium, $\varepsilon + \; > f(n_s{}_s + (n_w - 1){}_w; {}_w)$.

Corollary 2

By the assumption, when $f(y_i; y_j)$ is homogeneous of degree 0, so that $f(n_s \ \ _s + (n_w \ \ 1) \ \ _w; (n_s \ \ 1) \ \ _s + n_w \ \ _w) = f(\frac{n_s \ \ _s + (n_w \ \ 1) \ \ _w}{(n_s \ \ 1) \ \ _s + n_w \ \ _w}; 1) = f(\frac{n_s a + n_w \ \ 1}{(n_s \ \ 1)a + n_w}; 1)$, where $a = \frac{\ \ _s}{\ \ _w}$.

(i) If $\frac{n_s a + n_w \ \ 1}{(n_s \ \ 1)a + n_w}$ increases, $f(\frac{n_s a + n_w \ \ 1}{(n_s \ \ 1)a + n_w}; 1)$ increases by assumption. Let x denote $\frac{n_s a + n_w \ \ 1}{(n_s \ \ 1)a + n_w}$ here.

$$\frac{@x}{@a} = \frac{n_s + n_w \ \ 1}{(an_s \ \ a + n_w)^2} > 0$$

$$\frac{@x}{@n_s} = \frac{(a \ \ 1)a}{(an_s \ \ a + n_w)^2} < 0$$

$$\frac{@x}{@n_w} = \frac{1 \ \ a}{(an_s \ \ a + n_w)^2} < 0$$

When Utopia network is the unique equilibrium, it is trivial to determine the directions of $f^u = f(n_s \ \ _s + (n_w \ \ 1) \ \ _w; \ \ _w)$ regarding a, n_s, and n_w.

(ii) $n_w = n \ \ n_s$, so $\frac{n_s a + n_w \ \ 1}{(n_s \ \ 1)a + n_w} = \frac{n_s a + n \ \ n_s \ \ 1}{(n_s \ \ 1)a + n \ \ n_s}$. Let x denote $\frac{n_s a + n \ \ n_s \ \ 1}{(n_s \ \ 1)a + n \ \ n_s}$. Then

$$\frac{@x}{@a} = \frac{(a \ \ 1)^2}{((a \ \ 1)n_s \ \ a + n)^2} < 0:$$

It is also trivial to prove the uniqueness condition because $\ \ _s > \ \ _w$.

Corollary 4 and 5

First, when $\ \ ! 1 \ \ $, every $f(y_i; y_j) ! \ \ \frac{1}{2}$ when $y_i > y_j$. Then, $f^e = f^u ! \ \ \frac{1}{2}$, too. Thus, following Theorem 1, only for $" + \ \ \ \ \frac{1}{2}$, Utopia network is the unique Nash equilibrium. Second, when $\ \ ! \ \ 0$; every $f(y_i; y_j) ! \ \ 0$. Thus, $f^e = f^u ! \ \ 0$, too. Hence, for any $" + \ \ > \ \ 0$, Utopia network is the unique Nash equilibrium

Possible network con gurations with heterogeneous players in equilibrium

If the network con gurations are identical to the case with homogeneous players, the they are marked with an asterisk.

Lemma 2

By de nition of Nash equilibrium, complete strong dominant positive assortative matching g is a Nash equilibrium if and only if $u_i(g_i; g_{\ \ i}) \ \ u_i(g_i^0; g_{\ \ i})$ for every deviation $g_i^0 2 G_i$, for

	3 strong, 1 weak	1 strong, 3 weak	2 strong, 2 weak
Utopia	*	*	*
Positive assortative Matching	*	*	
Disassortative matching	*	*	* *

all $i \in N$. The strategy profile g for complete strong dominant positive assortative matching is as follows. (i) If player i's type is strong, $g_{ij} = 1$ for $j \in N_S \setminus \{i\}$, and $g_{ij} = -1$ for $j \in N_W$. (ii) If player i's type is weak, $g_{ij} = 1$ for all $j \in N \setminus \{k\}$.

There are four kinds of deviations from the complete strong dominant positive assortative matching. First, a strong player changes his positive links toward the other strong players to the negative links. Second, a strong player changes his negative links toward weak players to the positive links. Third, a strong player changes his positive links toward the other strong players to the negative links and changes his negative links toward weak players to the positive links. Lastly, a weak player changes his positive links toward the other players to the negative links.

First, the deviation $g + \sum_{j \in J} g_{ij}^-$ where $i \in N_S$ and $J \subseteq N_S \setminus \{i\}$ is unprofitable by Lemma 3. Second, the deviation $g + \sum_{j \in J} g_{ij}^+$ where $i \in N_S$ and $J \subseteq N_W$ is unprofitable when $u_i(g) \geq u_i(g + g_{ij})$ where $j \in N_W$ by Lemma 4. Third, the deviation $g + \sum_{k \in K} g_{ik}^- + \sum_{j \in N_W} g_{ij}^+$ where $i \in N_s$ and $K \subseteq N_s$ is unprofitable when $(n_s - 1)\alpha_s \geq n_w \alpha_w$ by Lemma 5. If $(n_s - 1)\alpha_s < n_w \alpha_w$, each deviation $g + \sum_{k \in K} g_{ik}^- + \sum_{j \in N_W} g_{ij}^+$ should be compared with the suggested strategy g whether it is profitable or not. Lastly, each weak player does not have any incentive to deviate from the suggested strategy. When a weak player changes his positive links to strong players, it cannot change the undirected links between the weak player and the strong players, but this deviation brings about additional conflict cost ε. If the weak player changes his positive links to other weak players, it is unpro table by Lemma 3.

Theorem 2

The condition that all Nash equilibria are SPM

Suppose a is large enough and $f(n_s; n_s \quad 1) > " + \quad$. Then CSPM is a Nash equilibrium, and any other network con guration except for SPM cannot be a Nash equilibrium. We will show the proof that CSPM is a Nash equilibrium in the next part after this part that there is no other equilibrium than SPM.

The network con gurations which are not SPM can be classi ed into three categories: (i) disassortative matching, (ii) other positive assortative matchings which are not SPMs (there exist weak players whose network strength is higher than some strong players' network strength.), and and (iii) Utopia network. I do not consider the war of all against all network, where there are only negative links. It is because this network is always not a Nash equilibrium when there are more than two players.

First, if a is large enough, any disassortative matching cannot be a Nash equilibrium. To show it, suppose there are K cliques $C_1; C_2; ::::; C_K$ in a disassortatvie matching g. By de nition of disassortative matching, there exists a clique C_l consisting of both strong and weak players. Suppose there are s strong players and w weak players in C_l, and players in this clique C_l has the C_lth strongest network strength in g. For a strong player i in this clique C_l, his utility is as follows.

$$u_i(g) = \sum_{j \, 2 \, \sum_{k=1}^{l \quad 1} C_k} (f(y_i; y_j) \quad ") + \sum_{j \, 2 \, \sum_{k=l+1}^{K} C_k} (f(y_i; y_j) \quad (" + \quad))$$

where $y_i = s \; s + w \; w$. Note that $f(y_i; y_j) < 0$ for $j \; 2 \; \sum_{k=1}^{l \quad 1} C_k$ and $f(y_i; y_j) > 0$ for $j \; 2 \; \sum_{k=l+1}^{K} C_k$. Now, let's consider a deviation strategy $g + g_{il}$ where player l is a weak player in the same clique C_l. The payo from the deviating strategy is

$$\begin{aligned} u_i(g + g_{il}) = & \sum_{j \, 2 \, \sum_{k=1}^{l \quad 1} C_k} (f(y_i \quad w; y_j) \quad ") + \sum_{j \, 2 \, \sum_{k=l+1}^{K} C_k} (f(y_i \quad w; y_j) \quad (" + \quad)) \\ & + f(y_i \quad w; y_i \quad s) \quad (" + \quad) \end{aligned}$$

To compare the sizes of $u(g)$ and $u_i(g + g_{il})$, let's substract $u(g)$ from $u_i(g + g_{il})$.

$$u_i(g + g_{il}) \quad u_i(g) = f(y_i \quad {}_w; y_i \quad {}_s) \quad ('' + \quad) \sum_{j \in \bigcup_{k=1}^{l-1} C_k} (f(y_i; y_j) \quad (f(y_i \quad {}_w; y_j)) \sum_{j \in \bigcup_{k=l+1}^{K} C_k} (f(y_i; y_j) \quad f(y_i \quad {}_w; y_j))$$

Using the property of f such that f is homogeneous of degree zero and $f(y_i; y_j) + f(y_j; y_i) = 0$,

$$u_i(g + g_{il}) \quad u_i(g) = f(y_i \quad {}_w; y_i \quad {}_s) \quad ('' + \quad) + \sum_{j \in \bigcup_{k=1}^{l-1} C_k} (f(\frac{y_j}{y_i}; 1) \quad (f(\frac{y_j}{y_i \quad {}_w}; 1)) \sum_{j \in \bigcup_{k=l+1}^{K} C_k} (f(\frac{y_i}{y_j}; 1) \quad (f(\frac{y_i \quad {}_w}{y_j}; 1))$$

If $a = \frac{s}{w}$ is large enoguh, $f(y_i \quad {}_w; y_i \quad {}_s) = f(\frac{sa + w \quad 1}{(s \quad 1)a + w}; 1) \; ! \quad f(\frac{s}{s \quad 1}; 1)$ while $f(\frac{y_j}{y_i}; 1)$ $f(\frac{y_j}{y_i \quad {}_w}; 1) \; ! \quad 0$ and $f(\frac{y_i}{y_j}; 1) \quad f(\frac{y_i \quad {}_w}{y_j}; 1) \; ! \quad 0$. $f(s; s \quad 1) \quad f(n_s; n_s \quad 1)$, and by the assumption, $f(n_s; n \quad 1) > '' + \quad$. Thus, $f(\frac{s}{s \quad 1}; 1) > '' + \quad$, and the disassortative matching g is not a Nash equilibrium.

Second, if a^a is large enough, any positive assortative matching which is not SPM cannot be a Nash equilibrium. A network con guration, which is PM but not SPM, has at least one strong player i whose network strength y_i and one weak player j whose network strength y_j. If a^a is large enough, y_i is always larger than y_j such that $y_i < y_j$. So it is a contradiction.

Lastly, Utopia network is not a Nash equilibrium if a is large enough and if $(n_s; n_s \quad 1) >$ $'' + \quad$. In Theorem 1, Utopia is a Nash equilibrium if $'' + \quad f^e = f(n_s \; {}_s + (n_w \quad 1) \; {}_w; (n_s$ $1) \; {}_s + n_w \; {}_w)$. As $a \; ! \; 1 \quad, f^e \; ! \quad f(n_s; n_s \quad 1)$. Therefore, when a is large enough and $f(n_s; n_s \quad 1) > '' + \quad$, Utopia network cannot be a Nash equilibrium.

If and only if condition that CSPM is a Nash equilibrium

If a is large enough, then Lemma 2 is satis ed given $'' + \quad < f \quad (n \quad 1; 1)$ Suppose a is su ciently large. Then the condition $(n_s \quad 1)a \quad n_w$ is satis ed. Let us check the condition $u(g) \quad u(g + g_{ij}^{+})$. The condition can be transformed as follow.

$$f(n_s a; n_w) > (n_w \quad 1)(f(n_s a + 1; n_w) \quad f(n_s a; n_w)) + '' + \quad :$$

If $a ! 1$, $f(n_s a; n_w) !$ f and $f(n_s a + 1; n_w)$ $f(n_s a; n_w) !$ 0. Therefore, as long as $" +$ $<$ f, there exists a satisfying the condition in Lemma 2.

Corollary 6

Using Lemma 2, for complete strong dominant positive assortative matching to be a Nash equilibrium, the following two conditions should be satis ed for $0 < d < n_s$.

$$\text{Condition (i)}: n_w \frac{1}{2}\frac{(n_s a) \quad n_w}{(n_s a) \quad + n_w} \quad n_w(" + \quad) \quad (n_w \quad 1)\frac{1}{2}\frac{(n_s a + 1) \quad n_w}{(n_s a + 1) \quad + n_w} \quad (n_w \quad 1)(" + \quad)$$

$$\text{Condition (ii)}: n_w \frac{1}{2}\frac{(n_s a) \quad n_w}{(n_s a) \quad + n_w} \quad n_w(" + \quad) \quad d\frac{1}{2}\frac{((n_s \quad d)a + n_w) \quad ((n_s \quad 1)a)}{((n_s \quad d)a + n_w) \quad + ((n_s \quad 1)a)} \quad d(" + \quad)$$

Every CSPM satis es $n_s a > n_w$, so $(n_s a + 1) > n_w$. Next, if $(n_s \quad d)a + n_w > (n_s \quad 1)a$, then $\frac{1}{2}\frac{((n_s \quad d)a + n_w) \quad ((n_s \quad 1)a)}{((n_s \quad d)a + n_w) \quad + ((n_s \quad 1)a)}$ converge to $\frac{1}{2}$. $d < da$ $(n_s \quad 1)a < n_w$, so condition (ii) is satis ed for $" +$ $<$ $\frac{1}{2}$. If $(n_s \quad d)a + n_w$ $(n_s \quad 1)a$, RHS of condition (ii) is always less than zero. Hence, condition (ii) is also satis ed, too.

Thus, if goes to in nity, $\frac{1}{2}\frac{(n_s a) \quad n_w}{(n_s a) \quad + n_w}$, $\frac{1}{2}\frac{(n_s a + 1) \quad n_w}{(n_s a + 1) \quad + n_w}$, and $\frac{1}{2}\frac{((n_s \quad d)a + n_w) \quad ((n_s \quad 1)a)}{((n_s \quad d)a + n_w) \quad + ((n_s \quad 1)a)}$ converge to $\frac{1}{2}$. $n_w > n_w$ 1 and $d < da$ $(n_s \quad 1)a < n_w$, so the condition (4) and (4) are always satis ed if $" +$ $<$ $\frac{1}{2}$.

Proposition 2

In this proof, we make proof by contradiction. We only show a condition for each weak player does not have an incentive to deviate from the complete weak dominant positive assortative matching g. It is because each strong type player decreases his/her utility when he/she extends any negative attitudes.

First, we show there exists a value making g a Nash equilibrium given n_w and n_s. Suppose $_s = \ _w +$ where > 0 is small enough. For a weak player i, g and any deviation $g^0 = g + \sum_{j 2 J} g^+_{ij} + \sum_{k 2 K} g_{ik}$ where K $N_w \, n \, f i g$ and J N_s gives utilities are as follows.

$$u_i(g) = n_s f(n_w \ _w; n_s \ _s) \quad n_s(" + \quad) \tag{2}$$

$$u_i(g^0) = (n_s \ j \ J j) f((n_w \ j \ K j) \ _w + jK j \ _s; n_s \ _s) +$$

$$jK j f((n_w \ j \ K j) \ _w + jK j \ _s; (n_w \quad 1) \ _w) \quad (n_s \ j \ J j + jK j)(" + \quad) \tag{3}$$

$$< (n_s \ j \ J j + jK j) f((n_w \ j \ K j) \ _w + jK j \ _s; n_s \ _s) \quad (n_s \ j \ J j + jK j)(" + \quad) \tag{4}$$

(4) > (3) because $n_w \quad 1 > n_s$ and is small enough. Also, (2) > (4). Note that (4) is $u(g^{00})$ where $g^{00} = g + \sum_{l \in L} g^+_{il}$ where $|L| = |J| \; K$ and $L \quad N_s$. By lemma 4, if $u_i(g) \quad u_i(g + g^+_{ij})$ for a $j \in N_s$, $u_i(g) \quad u_i(g^{00})$. $u_i(g)$ is always greater than $u_i(g + g^+_{ij})$ for small enough as follows. By the assumptions, $\frac{\partial^2 f}{\partial_i^2} < 0$, $n_w > 2n_s \quad 1$, and $" + $ is small enough. Then

$$n_w \;\; _w > (2n_s \quad 1) \;\; _s$$

$$n_w \;\; _w \quad n_s \;\; _s > (n_s \quad 1) \;\; _s$$

$$f(n_w \;\; _w; n_s \;\; _s) \quad f(n_s \;\; _s; n_s \;\; _s) \quad (" + \quad) > (n_s \quad 1)(f(n_w \;\; _w + \;\; _s; n_s \;\; _s) \quad f(n_w \;\; _w; n_s \;\; _s)) \tag{5}$$

The last line holds, because $\frac{\partial^2 f}{\partial_i^2} < 0$. Adding small enough $" +$ and rearranging the last line,

$$n_s f(n_w \;\; _w; n_s \;\; _s) \quad n_s(" + \quad) > (n_s \quad 1) f(n_w \;\; _w + \;\; _s; n_s \;\; _s) \quad (n_s \quad 1)(" + \quad)$$

Therefore, if a is close to 1 enough and $n_w > 2n_s \quad 1$, complete weak dominant positive assortative matching is a Nash equilibrium.

Corollary 8

First, let us consider the case when $" + \quad < \frac{1}{2}$: There is a weak player i's deviation $g^0_i = g_i + \sum_{j \in N_s} g^+_{ij} + \sum_{k \in K} g_{ik}$ where $K \quad N_w \setminus \{i\}$ and $|K| = n_s + 1$, which is more pro table for i when $\; ! \; 1$. After the deviation, $n_i(g^0) = (n_w \quad n_s \quad 1) \;\; _w + n_s \;\; _s$ and $n_k(g^0) = n_w \quad 1$. Thus $n_i(g^0) > n_k(g^0)$. Player i's utility from g^0 is approximately

$$u(g^0) \quad (n_s + 1)(\frac{1}{2} \quad (" + \quad));$$

and the utility from CWDPAM is approximately

$$u(g) \quad n_s(\frac{1}{2} \quad (" + \quad)):$$

Therefore, the deviation is always more pro table than CWPM.

If $" + \quad \frac{1}{2}$, any extraction is not pro table because this cost is higher than the upper bound of the possible extraction. Therefore, CWPM cannot be a Nash equilibrium.

Appendix B. Four players example: The condition for each con guration to be a Nash equilibrium

Homogeneous players

There are two network con guration, which can be Nash equilibria. The rst case is Utopia network. According to Remark 3, it is always a Nash equilibrium for any" + , and it is unique when" + > f (3; 1). The second case is a network where three players are a friend to each other, and they extend negative links to the other player. It is a Nash equilibrium when " + f (3; 1) by Lemma 1.

Except for these two con gurations, any other network con gurations cannot be Nash equilibria by Remark 1. In the other con gurations, there always exist players whose network strengths are the same. It violates Remark 1.

The extraction value f in g is zero because every player has the same aggregate strength 2 , but the con ict cost is positive. In the fourth network, the players constituting the small cliques have an incentive to deviate by extending a positive link to each other for the same reason as in the third case. In the last network, all players have the same aggregate strength , so any player extending a negative link has an incentive to change it to a positive link for the same reason as in the third case. Therefore, each of the three networks cannot be a Nash equilibrium.

Two types of players

Network con gurations in Figure 7, 8, and 9 can be Nash equilibria depending on the conditions. The condition for each con guration to be a Nash equilibrium is speci ed in the caption.

(a) " + f (3 s; 2 s + w) (b) " + f (s + 2 w; 3 w) (c) " + f (2 s + w; s + 2 w)

Figure 7: Utopia networks

Appendix C. Simulation

3.1.1 Utopia networks: Homogeneous players' model

The blue part is a region of the parameters + , and n for the Utopia network to be the unique Nash equilibrium. The yellow part represents the boundary, and it is corresponding to $\underline{c}^{uh}$. Note that $\frac{1}{2}$ is the maximum value of $h(n_i; n_j;)$. When is larger than 2 and n is larger than 6, $\underline{c}^{uh}$ is already close to $\frac{1}{2}$.

3.2.1 Utopia networks: The case of two types of players

Figure 11 compares the conditions for Utopia networks to be a Nash equilibrium and to be the unique Nash equilibrium in the example of four players. Figure 12 presents variations of ranges of the parameters supporting Utopia network as a Nash equilibrium with respect to when there are two strong players and two weak players. It also indicates that the conflict cost should increase and a should decrease as increases to maintain the Nash equilibrium.

3.2.2 Positive assortative matching

Figure 13 shows the ranges for each positive assortative matching of Figure 5 (a), (b), and (c) to be a Nash equilibrium. It is derived by using the contest success function h for the extraction function f. Each graph is a special case of Corollary 7. Figure 13(a) shows the difference with respect to the number of players when $= 1$. Figure 13(b) describes a three-dimensional range for the configuration with two strong players and two weak players to be a Nash equilibrium. Figure 13(c) is a two dimensional expression of Figure 13(b). The figures indicate the following. First, if a is sufficiently large, then this CSPM is a Nash equilibrium. Second, If the number of strong players increases, less a can make CSPM in equilibrium. Third, if the conflict cost is sufficiently high, then CSPM can not be a Nash equilibrium. It is consistent with Proposition 3 mentioning the uniqueness of the Nash equilibrium Utopia network. Lastly, as the number of strong players increases or increases, CSPM can be a Nash equilibrium with the higher conflict cost.

Figure 13b and 13c also shows consistent observations with Corollary 6. It presents the variation in respect of on the range of parameters satisfying the condition for the Nash equilibrium strong dominant positive assortative matching when there are two strong and two weak players. In this diagram, as increases (decreases), the range expands (shrinks).

As mentioned above, Figure 13a demonstrates variations in respect to the number of players on the range of parameters satisfying the condition. The figure indicates that the range expands (shrinks) when there are more (less) strong players and less (more) weak players.

Figure 14 demonstrates variations of the range of parameters satisfying the conditions for the Nash equilibrium weak dominant positive assortative matching when there are one strong and three weak players. These diagrams in Figure 14 give implications consistent with Proposition 2, Corollary 8 and Remark 7. When increases, the range of f+ increases and the range of a decreases. For example, in the extreme case such as 100, no a satis es the condition for this Nash equilibrium con guration. Lastly, there is an upper bound of + for the con guration in equilibrium. It is consistent with Proposition 1 in respect of uniqueness of the Nash equilibrium Utopia network.

(a) $f(3\ _s;\ _w)\quad " +$

(b) $f(2\ _s;\ _s) + f(2\ _s;\ _w)\quad 2(" + \ ),$
$f(2\ _s;\ _s)\quad (f(3\ _s;\ _w)\quad f(2\ _s;\ _w))\quad " + \ ,$
$2(f(2\ _s;\ _s + \ _w)\quad f(2\ _s;\ _s)) + f(\ _s;\ _w)\quad 0$

(c) $f(\ _s;\ _w)\quad " + \ ;$
$2(f(\ _s + \ _w; 3\ _w)\quad f(\ _s; 3\ _w))$
$f(\ _s; 3\ _w)\quad "$

(d) $f(3\ _w;\ _s)\quad " + \ ,$
$3f(\ _w;\ _s)\quad f(2\ _w + \ _s; 2\ _w);$
$f(3\ _w;\ _s)\quad 2f(\ _w + \ _s; 2\ _w)\quad "$

(e) $2f(\ _s; 2\ _w) + f(\ _s;\ _w)\quad " + \ ;$
$f(\ _s; 2\ _w)\quad (f(\ _s + \ _w; 2\ _w)\quad f(\ _s; 2\ _w))$
$+ (f(\ _s + \ _w;\ _w)\quad f(\ _s;\ _w)) + " + \ ;$
$2f(\ _s; 2\ _w)$
$(f(\ _s + 2\ _w;\ _w)\quad f(\ _s;\ _w)) + 2(" + \ );$
$f(2\ _w;\ _w) + f(\ _s; 3\ _w)\quad f(\ _s; 2\ _w)\quad " +$

(f) $f(2\ _s; 2\ _w)\quad " +$
$2f(2\ _s; 2\ _w)\quad f(2\ _s + \ _w; 2\ _w) + " +$

Figure 8: Positive assortative matching

(a) $f(2\lambda_S + \lambda_W, \lambda_S) \geq \varepsilon + \kappa$
$f(3\lambda_S, \lambda_S + \lambda_W) \leq f(2\lambda_S + \lambda_W, \lambda_S)$

(b) $f(\lambda_S + 2\lambda_W, \lambda_W) \geq \varepsilon + \kappa$,
$f(\lambda_S + \lambda_W, 2\lambda_W) \leq$
$f(\lambda_S + 2\lambda_W, \lambda_W) - f(\lambda_S + \lambda_W, \lambda_W)) + \varepsilon + \kappa$

(c) $f(\lambda_S + 2\lambda_W, \lambda_S) \geq \varepsilon + \kappa$,
$f(2\lambda_S + \lambda_W, 2\lambda_W) \leq f(\lambda_S + 2\lambda_W, \lambda_S)$,
$f(\lambda_S + \lambda_W, 2\lambda_W) \leq$
$f(\lambda_S + 2\lambda_W, \lambda_S) - f(\lambda_S + \lambda_W, \lambda_S) + \varepsilon + \kappa$,
$f(\lambda_S + 2\lambda_W, \lambda_S) + \varepsilon + \kappa \geq 2f(\lambda_S, 2\lambda_W)$

(d) $f(2\lambda_S + \lambda_W, \lambda_W) \geq \varepsilon + \kappa$,
$f(2\lambda_S, \lambda_S + \lambda_W) \leq$
$f(2\lambda_S + \lambda_W, \lambda_W) - f(2\lambda_S, \lambda_W) + \varepsilon + \kappa$

Figure 9: Disassortative matching

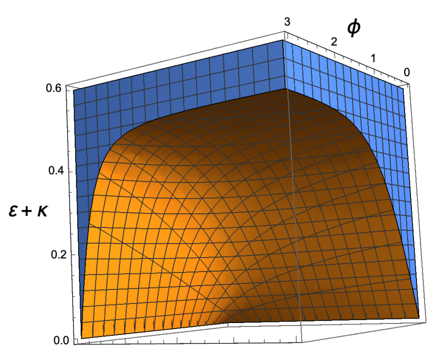
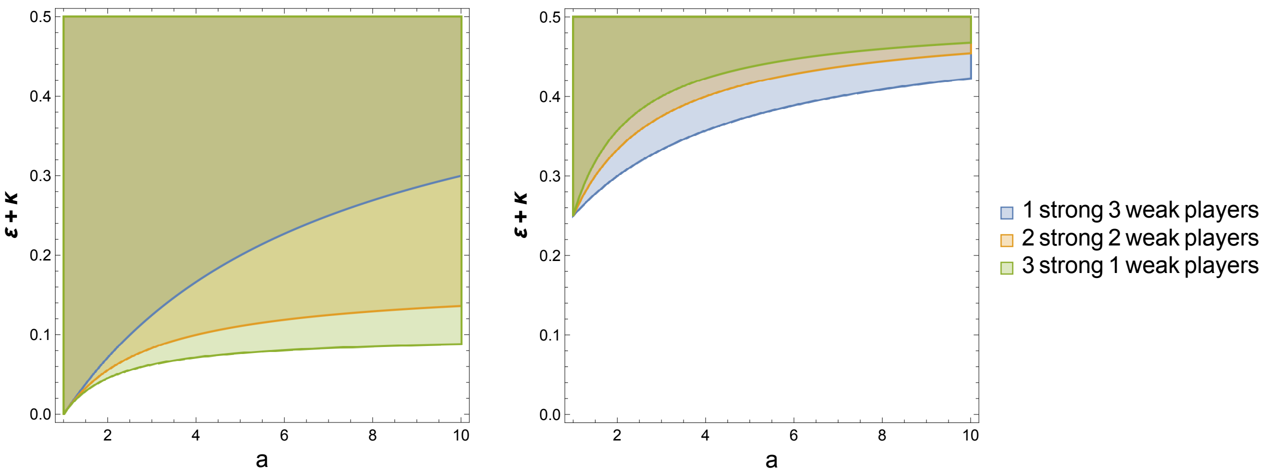


Figure 10: Region of the parameters for the Utopia networks as the unique Nash equilibrium



(a) Existence condition (b) Uniqueness condition

Figure 11: Variation in ranges of the parameters with respect to the number of players when = 1 for the Utopia network

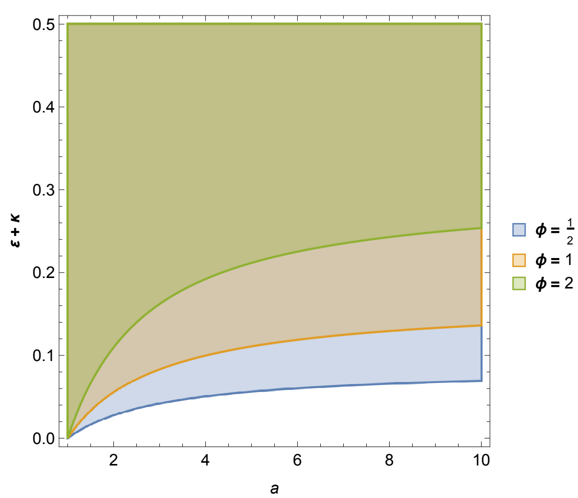
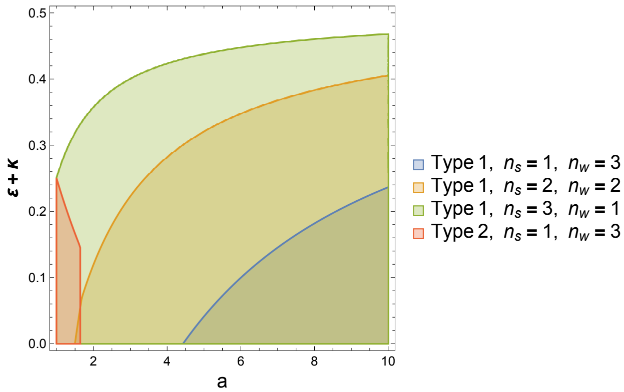
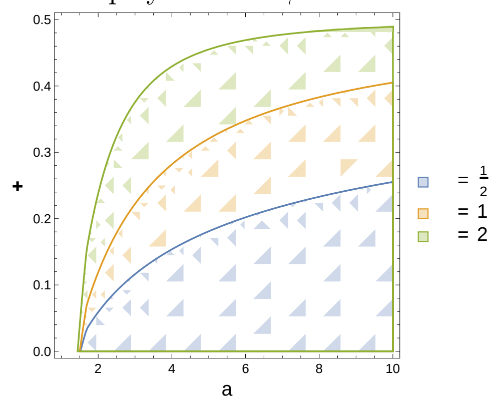


Figure 12: Variation of uniqueness condition for Utopia network with respect to with 2 strong and 2 weak players



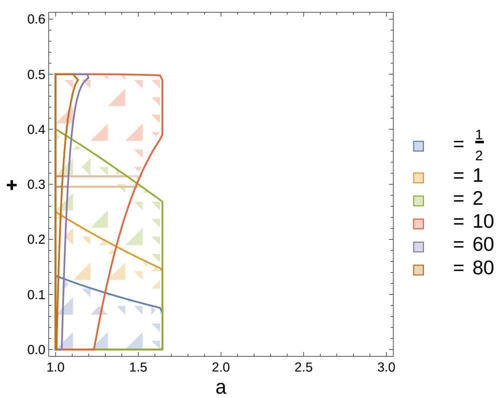
(a) Variation with respect to the number of players when $\phi = 1$



(b) CSPM, when $n_S = 2$ $n_W = 2$

(c) Variation with respect to ϕ when $n_S = 2$ & $n_W = 2$ using CSPM

Figure 13: Ranges of the parameters supporting the positive assortative matching



(a) Variation with respect to ϕ when $n_S = 1$ & $n_W = 3$ using the weak dominant

(b) Using the weak dominant, n_S=1 n_W=3

Figure 14: Changes in Range of the parameters supporting the Nash equilibrium CWPM by