Selling to Wishful Thinkers*

Tommy Chao Hung Chan† Matthew Kovach‡
(Job Market Paper)

December 1, 2023

Click Here for the Most Recent Version of the Paper

Abstract: Empirical evidence consistently shows that individuals often skew their probability assessments optimistically, a phenomenon known as wishful thinking (WT). This paper investigates the impact of WT on bidder behavior in private value auctions. Critically, under WT beliefs about others are endogenous to the mechanism. We find that wishful thinking induces underbidding in traditional first-price sealed-bid auctions, corroborating recent experimental findings. Importantly, behavior in second-price auctions is unchanged, and so revenue equivalence does not hold under WT. Next, we derive the optimal mechanism under WT and find that (i) the optimal mechanism is a variant of the sad-loser auction, as first proposed by Riley and Samuelson (1981), and (ii) (expected) seller revenue is increasing in the degree of wishful thinking exhibited. These insights illustrate the importance of understanding bidders' belief biases in auction design.

Keywords: Auctions; Wishful Thinking; Optimism; Sad-Loser Auction
JEL:

*We would like to thank Rohit Lamba, Debasis Mishra, Antonio Rosato, Sudipta Sarangi, as well as participants at the 12th Conference on Economic Design (Padova), 17th Conference on Economic Growth and Development (ISI Delhi) and Midwest Economic Theory 2022 (Notre Dame) for comments and suggestions.

†Virginia Tech, chaohungc@vt.edu.
‡Purdue University, mlkovach@purdue.edu

1 Introduction

Beliefs about economic variables and the behavior of others play critical roles in many economic settings. For example, an investor's decision whether or not to purchase shares in a firm depends on her beliefs about the firm's future profits. Likewise, individuals decide on their insurance levels based on their beliefs about the likelihood of accidents or health complications. It is common to assume in economic models that beliefs are "correctly calibrated," or decision makers exhibit rational expectations, in contrast to the myriad of biases observed in experimental economics.

This paper explores the implications of a particular type of belief bias, wishful thinking (WT), in auctions with independent, private values. WT is a form of optimism where the desirability of an outcome influences its perceived probability. In an auction setting, a wishful thinker misperceives the distribution of the other's values and hence she also misperceives the distribution of payoffs from the mechanism. To model WT, we adapt Kovach (2020) to an auction setting. A bidder's beliefs are "twisted" via a distortion function that transforms the true distribution over values into a subjective distribution that "shifts" probability mass from bad outcomes to good outcomes. In short, this means that a wishful bidder puts excessive probability on other bidders realizing low valuations; she believes she is more likely to win and pay a lower price conditional on winning than she should.

We start our analysis by studying the impact of WT in common auction formats. First, we show that WT induces underbidding in the standard first-price auction because bidders overestimate their chances of winning given their bid. Notably, this does not arise in the second-price auction, since the (weakly) dominant strategy is independent of a bidder's beliefs about others. Therefore, revenue equivalence does not hold for wishful bidders.

The mechanism design literature primarily assumes (subjective) expected utility preferences, which allows for additive separation of agents' expected utility into expected allocation and expected payment. Using the envelope theorem, the expected revenue of an incentive-compatible mechanism becomes a function of the expected allocation for each type (e.g., Myerson-type characterization). When beliefs are distorted due to wishful thinking expectations are not consistently additive, complicating the application of Myerson-type characterization.

Consider a scenario with two risk-neutral bidders. One is susceptible to WT, while the other is a standard (subjective) expected utility agent. For the bidder not influenced by WT, all mechanisms that yield the same expected payment and expected allocation are considered equivalent. In this case, the seller can disregard the actual distribution of payment and allocation, focusing solely on their expected values.

However, this principle does not apply to the bidder affected by WT. Let's consider a situation where both bidders are presented with two different mechanisms, each offering the same expected allocation and expected payment from the perspective of the seller. One mechanism provides winning probability and payment unconditionally, irrespective of others' reports, while the other is conditional on those reports. The wishful thinker will typically prefer the mechanism with outcomes contingent on others' reports. This preference arises because wishful thinkers tend to adjust their beliefs to favor scenarios that yield more favorable results. When outcomes hinge on factors with subjective uncertainty, like others' reports, there is room for bidders to engage in wishful thinking and imagine more favorable outcomes. This leads them to overestimate the actual utility value of a particular bid (i.e., because they overestimate their chances of winning).

From the seller's perspective, both mechanisms are equally feasible. However, the one with outcomes conditioned on others' reports has the potential to generate more surplus that can be extracted, provided all bidders report truthfully. Consequently, the mechanism that conditions outcomes on others' types will consistently yield higher revenue.

Further complicating the problem is the seller's authority over the distribution of outcomes. This control plays a pivotal role in shaping the extent to which the bidder is influenced by WT, rendering the problem intricate. Presently, the seller bears the responsibility of crafting the intricate distribution of payment and allocation. This introduces non-additivity to the expectation operator, as beliefs evolve alongside the distribution of outcomes. Intuitively, WT induces a "positive feedback" loop between behavior and beliefs. Navigating this complexity is achieved through the application of the converse envelope theorem, as proposed by (Sinander, 2022).

Our main results reveal that sellers can exploit WT-affected bidders by employing a variant of the sad-loser auction, as first introduced by (Riley and Samuelson, 1981). Wishful thinkers typically underestimate their chances of being the "sad loser," making this auction strategy particularly effective in leveraging their behavior.

The remainder of the paper is structured as follows. Section 2 reviews the related literature. Section 3 provides a simple motivational example to illustrate how wishful thinking can influence bidders' behavior. In Section 4, we formalize the concept and introduce relevant techniques that we employ to address the problem. Section 5 and 6 delve into the specific examination of two types of wishful thinking distortions and their impact on the action design problem. Section 7 presents a discussion of the results and their implications

2 Related Literature

2.1 Wishful Thinking

Wishful Thinking (WT) occurs when an event's desirability influences its perceived probability. The psychological literature has long acknowledged this bias (Granberg and Brent, 1983; Hogarth, 1987; Cohen, 1992). The psychology literature suggests that WT is context-specific, and is more frequently observed in competitive situations and those involving subjective probabilities. (Krizan and Windschitl, 2007).

More recently, wishful thinking has been gaining traction in economics, with experimental evidence in Mayraz (2011) and Engelmann et al. (2023) and an axiomatic model developed by Kovach (2020). In Mayraz (2011), subjects were assigned roles as either *farmers* or *bakers*, and were tasked with predicting wheat prices from data. If subjects form rational beliefs, then they should make similar predictions across roles because they observed the same information. The results, however, show that subjects' beliefs are influenced by their role, in line with wishful thinking. Engelmann et al. (2023) finds evidence for wishful thinking when subjects are faced with anxiety due to future discomfort or losses. They also find that Wishful thinking is more pronounced with ambiguous information.

2.2 Mechanisms with non-SEU bidders

Our paper contributes to the growing literature on mechanisms designed under bounded rationality or behavioral bidders.

Most mechanism design with non-SEU bidders has focused on two specific preference theories: Maxmin Expected Utility(Gilboa and Schmeidler, 1989) and Cumulative Prospect Theory (Tversky and Kahneman, 1992). For example, a recent addition to the literature on loss aversion in auctions includes Rosato (2023). In this paper, he considers sequential auction design and shows that loss aversion induces history-dependence and a discouragement effect.

Other forms of bounded rationality have also been considered. For instance, Gagnon-Bartsch et al. (2021) considered the impact of projection bias in auctions. With independent private values, such bidders overbid in first-price auctions, but not in second-price ones. The focus of the paper, however, is on common value auctions. They show that in such settings second-price auctions are less efficient than first-price auctions.

2.3 The Sad-Loser Auctions

Our main results show that the revenue-maximizing auction for WT bidders is a variant of the sad-loser auction, first introduced by Riley and Samuelson (1981). This auction type is characterized by a system where only the losers pay, while the winners receive goods free of charge.

The sad-loser auction has been shown to maximize revenue in a variety of settings, including Tullock contests (Cohen and Sela, 2005; Matros, 2012; Minchuk, 2018) and when bidders are risk loving (e.g., bidders have exponential utility)(Nikolova et al., 2018).

3 Motivating Example: Wishful Thinking and Underbidding in FPA

Consider a private value auction with two bidders who exhibit Wishful Thinking (WT). In this setting, the bidders' valuations θ are independently drawn from a uniform distribution $U[0,1]$. WT is a bias in the bidders' beliefs in which they assign excessive probability to "better" outcomes. We start with a simple way to capture this idea and suppose that each bidder "shifts" their beliefs slightly toward the best-case scenario. In this example, the best case occurs when the other bidders do not value the good at all. We can therefore model this bias using the distorted belief $G^{\delta}(\theta) = (1-\delta)G(\theta) + \delta\mathbb{1}\{\theta = 0\}$, with δ capturing the degree of bias.

Our objective is to examine the implications of WT on bidder behavior in two common auction formats: the First-Price Auction (FPA) and the Second-Price Auction (SPA). FPA involves bidders submitting sealed bids, with the highest bidder winning the item and paying their bid. In SPA, the highest bidder wins but pays the price of the second-highest bid.

To illustrate the effects of WT, we analyze the symmetric Bayesian Nash equilibrium (BNE) in both FPA and SPA, assuming a limit case where the minimum bid approaches zero. Our focus is on the bidding strategies and resulting seller revenue in the presence of WT.

$\beta(\theta)$	SEU	WT
FPA	$\frac{\theta}{2}$	$\max\left\{\frac{\theta}{2} - \frac{\delta}{1-\delta}, 0\right\}$
SPA	θ	θ

Table 1: Bidding Strategies in FPA and SPA

Table 1 summarizes the bidding strategies in FPA and SPA for both WT bidders and bidders with Subjective Expected Utility (SEU). The strategies reflect the influence of WT on bidder behavior. We observe that WT bidders tend to underbid in FPA, adjusting their bids downward due to their belief that others' valuations are lower than standard agents would believe. Conversely, in SPA, WT bidders tend to reveal their true valuations, resulting in bidding strategies that align with SEU bidders.

These findings have important implications for auction outcomes and seller revenue. In FPA, the bias induced by WT leads to reduced seller revenue compared to auctions with SEU bidders.[1] However, in SPA, the bidding strategy of revealing the true type dominates, resulting in revenue equivalence between WT bidders and SEU bidders.

By examining the impact of WT on bidder behavior and auction outcomes, we shed light on the need for alternative mechanisms. One direction is to look at seller optimal mechanisms - can the seller exploit the bias to increase their profits? In the following sections, we delve into the manipulation of beliefs and the design of optimal mechanisms, aiming to maximize seller revenue and mitigate the impact of WT-induced underbidding.

4 Framework and Notation

This section presents the general framework and notation used throughout the paper. We consider a single good auction with n bidders subject to wishful thinking bias. The set of bidders is denoted as $N = 1, 2, ..., n$, with bidder i being the typical bidder. Each bidder i has a quasi-linear utility function defined as $u(q, t, \theta_i) = \theta_i q - t$, where q represents the probability of winning the good, t denotes the monetary payment (transfer to the mechanism), and θ_i represents bidder i's private valuation or type. The payment is positive and bounded by the budget constraint $\bar{t} > 1$. The valuations θ_i are drawn independently from a distribution F with a continuous density f over

[1]Previous experimental results found a tendency to overbid in first-price sealed-bid auctions, compared with the risk-neutral Nash Equilibrium (Kagel and Levin, 2008). The literature offered various explanations, such as risk aversion (Cox et al., 1988), regret aversion (Engelbrecht-Wiggans, 1989), and spiteful motives (Morgan et al., 2003). However, a recent study by Neugebauer and Selten (2006) suggests the overbidding behavior may be influenced by the information feedback process in FPA's standard setting. Our research builds upon this observation and proposes that Wishful Thinking (WT) may provide an explanation for underbidding in FPA.

the possible types $\Theta = [0, 1]$. The distribution of types for the bidders is identical and independent, denoted as $G = F^{N-1}$ over $\Theta_{-i} = \Theta^{N-1}$.

To describe uncertainty that depends on others' types, we introduce random variables defined on the probability space $(\Theta_{-i}, \mathcal{B}^{n-1}, G)$, where $\Theta_{-i} = [0, 1]^{n-1}$ and $\mathcal{B}^{n-1}$ is the Borel σ-algebra of $\mathbb{R}^{n-1}$ restricted to $[0, 1]^{n-1}$. Capital calligraphic letters, such as $\mathcal{Q}, \mathcal{P}, \mathcal{T}$, represent sets of real-valued random variables defined on Θ_{-i}, while boldface lowercase letters, such as $\mathbf{q}, \mathbf{p}, \mathbf{t}$, denote specific elements of these sets. Furthermore, boldface capital letters, such as $\mathbf{Q}, \mathbf{P}, \mathbf{T}$, represent functions that map types to random variables defined on Θ_{-i}. For example, $\mathbf{Q} : \Theta \to \mathcal{Q}$ implies $\mathbf{Q}(\theta)(\theta_{-i})$ can be denoted as $Q(\theta, \theta_{-i})$, where $Q : \Theta \times \Theta_{-i} \to [0, 1]$.

In the presence of wishful thinking bias, bidders' beliefs about the probability of winning and payment depend on others' types θ_{-i}. This introduces uncertainty for bidder i in the form of an uncertain state. For any state-dependent utility $\mathbf{u} : \Theta_{-i} \to \mathbb{R}$, the subjective expectation is denoted as $\mathbb{E}_{-i}\mathbf{u} = \int_{\theta_{-i} \in \Theta_{-i}} \mathbf{u}(\theta_{-i}) G(d\theta_{-i})$. With wishful thinking, beliefs are distorted towards favorable events. This is captured by a positive non-decreasing distortion function δ, and the distorted expectation becomes $\mathbb{E}^{\delta}_{-i}\mathbf{u} = \int_{\theta_{-i} \in \Theta_{-i}} \mathbf{u}(\theta_{-i}) \delta(\mathbf{u}(\theta_{-i})) G(d\theta_{-i})$. For any random variable $\mathbf{x}$ defined on $(\Theta_{-i}, \mathcal{B}^{n-1}, G)$, the distorted expectation $\mathbb{E}^{\delta}\mathbf{x}$ represents the expectation of $\mathbf{x}$ under the distribution G^{δ}, where $G^{\delta}(d\theta_{-i}) = \delta(\mathbf{x}(\theta_{-i})) G(d\theta_{-i})$. Further discussion on the distortion will be provided in the following subsection.

The seller aims to maximize the expected payment by selecting a direct mechanism. Each bidder i submits a report $\hat{\theta} i$ to the seller, and based on the profile of reports $\hat{\theta} = (\hat{\theta} 1, ..., \hat{\theta} N)$, the mechanism determines the probability of winning and the payment for each bidder, denoted as $Q_i(\hat{\theta}), T_i(\hat{\theta}) i \in N$. The mechanism is anonymous with respect to bidders, meaning that for bidder i and j, when $(\theta_i, \theta{-}i) = (\theta_j, \theta{-}j)$, we have $Q_i(\theta_i, \theta_{-i}) = Q_j(\theta_j, \theta_{-j})$ and $T_i(\theta_i, \theta_{-i}) = T_j(\theta_j, \theta_{-j})$.

To emphasize the uncertainty of other types of an agent, we treat the mechanism as a pair of random variables, $\mathbf{Q}$ and $\mathbf{T}$. Both take an agent's report as input and return a random variable defined on the probability space of others' reports. The schedule $\mathbf{Q}$ maps reports, θ_i, to a real random variable $\mathbf{Q}(\theta_i)$ that defines the winning probability conditional on others report θ_{-i}. Similarly $\mathbf{T}$ maps a report, θ_i, to a real random variable $\mathbf{T}(\theta)$ that the payment conditional on others report θ_{-i}. The set $\mathcal{Q}$ denotes the measurable functions with range $[0,1]$ defined on the probability of others' types, $\mathcal{Q} = \{\mathbf{q} : \Theta_{-i} \to [0,1] | \mathbf{q} \text{ is measurable}\}$; the set $\mathcal{T}$ denotes the measurable functions with range $[0,\bar{t}]$ defined on the probability of others' type, $\mathcal{T} = \{\mathbf{t} : \Theta_{-i} \to [0,\bar{t}] | \mathbf{t} \text{ is measurable}\}$. A mechanism is denoted as $(\mathbf{Q},\mathbf{T})$, $\mathbf{Q} : \Theta_i \to \mathcal{Q}$ and $\mathbf{T} : \Theta_i \to \mathcal{T}$. With mechanism $(\mathbf{Q},\mathbf{T})$, a type θ_i agent reporting $\hat{\theta}_i$ will have a random utility, $\mathbf{U}(\hat{\theta}_i, \theta_i | \mathbf{Q}, \mathbf{T}) = u(\mathbf{Q}(\hat{\theta}_i), \mathbf{T}(\hat{\theta}_i), \theta_i)$. When the mechanism is clear from the context, we will omit the notation $\mathbf{Q}, \mathbf{T}$.

The mechanism is feasible when the typical Bayesian Incentive Compatible (BIC), Individual Rationality (IR), and Plausibility (P) holds.

$$\forall \theta_i, \theta_i' \in \Theta, \theta_i' \neq \theta_i : \qquad \mathbb{E}^{\delta}_{-i}\{\mathbf{U}(\theta_i, \theta_i)\} \geq \mathbb{E}^{\delta}_{-i}\{\mathbf{U}(\theta_i', \theta_i)\}, \tag{BIC}$$

$$\forall \theta_i \in \Theta : \qquad \mathbb{E}^{\delta}_{-i}\{\mathbf{U}(\theta_i, \theta_i)\} \geq 0, \tag{IR}$$

$$\forall (\theta_1, ..., \theta_n) \in \Theta^n \qquad \sum_{j \in N} Q(\theta_j, \theta_{-j}) \leq 1, \text{ and } \forall i \in N : Q(\theta_i, \theta_{-i}) \geq 0, \tag{P}$$

where $Q(\theta_i, \theta_{-i}) = \mathbf{Q}(\theta_i)(\theta_{-i})$.

4.1 Incorporating Wishful Thinking

We first clarify how wishful thinking is modeled in this setting. In an auction, there are two sources of uncertainty. The randomness of others' reports $\hat{\theta}_{-i}$, and given a report the mechanism may further give a randomized allocation $Q(\hat{\theta}_i, \hat{\theta}_{-i}) \in (0, 1)$. We assume that WT only distorts a bidder's beliefs about others' reports, not the way the mechanism works.[2]

We study two forms of WT axiomatized in Kovach (2020): the best-case binary distortion and the consequential distortion. In both cases, real random variables (acts) (e.g., $\mathbf{x}$ defined on probability space (Ω, Σ, F)) are evaluated by the distorted expectation $\mathbb{E}^{\delta}\mathbf{x} = \mathbb{E}\mathbf{x}\delta(\mathbf{x})$. The distortion δ is an increasing function that captures wishful thinking. The best-case wishful thinker increases the probability of the best-case scenario by determining other cases' probability proportionally, while the consequential wishful thinker reweighs the probability density of each event by a distortion function and renormalizing afterward. The distortion function is increasing in utility to capture wishful thinking:

1. Best-case binary distortion:

$$\delta(\mathbf{U}(\hat{\theta}_i, \theta_i)(\theta_{-i})) = \begin{cases} 1 - \delta & \text{if } \theta_{-i} \notin \mathcal{B}(\hat{\theta}_i, \theta_i), \\ 1 - \delta + \delta \frac{1}{G(\mathcal{B}(\hat{\theta}_i, \theta_i))} & \text{if } \theta_{-i} \in \mathcal{B}(\hat{\theta}_i, \theta_i), \end{cases}$$

$\mathcal{B}(\hat{\theta}_i, \theta_i)$ is the θ_{-i}'s that give the maximum utility for type θ_i with report $\hat{\theta}_i$. $\delta \in (0, 1)$ is some constant.[3] Alternatively, one may think of the distorted belief

[2]This is in line with experimental evidence suggesting that wishful thinking arises more frequently in subjective or ambiguous environments (such as competitions), rather than games of objective chance (such as roulette). (Krizan and Windschitl, 2007).

[3]In its most general form, δ is not a constant but depends on the act (report) and type. As we focus on truthful reporting cases, it should depend on the type. We assume all types have the same level of distortion for simplicity here.

as $G^\delta = (1-\delta)G + \delta D$, a convex combination between correct belief G and some distortion belief D different from G. In this example, D is given by a uniform distribution over "best case types," which could be discrete. With this definition, we extend the best-case binary distortion to the cases of discrete best cases.

2. Consequential distortion:

$$\delta(\mathbf{U}(\hat{\theta}_i, \theta_i)) = \frac{v(\mathbf{U}(\hat{\theta}_i, \theta_i))}{\mathbb{E}_{-i} v(\mathbf{U}(\hat{\theta}_i, \theta_i))},$$

where the distortion function v is a continuous and increasing function.

Note that bidders have correct beliefs when they have no stake in the game (or choose a constant act). This reflects the nature of wishful thinking, which distorts "subjective" events optimistically but does not distort objective lotteries.

We only require that the distorted preference is monotonic with respect to First Order Stochastic Dominance (FOSD). Note that the utility function depends on the mechanism so that we denote the distortion as $\delta(\theta_{-i}|\hat{\theta}_i, \theta_i, Q, T) := \delta(\mathbf{U}(\hat{\theta}_i, \theta_i)(\theta_{-i}))$, and the partial derivative of it with respect to the truth type as $\delta_\theta(\theta_{-i}|\hat{\theta}_i, \theta_i, \mathbf{Q}, \mathbf{T}) = \frac{\partial \delta(\theta_{-i}|\hat{\theta}_i, \theta_i, \mathbf{Q}, \mathbf{T})}{\partial \theta_i}$. As a random variable defined on Θ_{-i} denoted as $\delta_\theta(\mathbf{U}(\hat{\theta}_i, \theta_i))$.

4.2 The Envelope Theorem and Its Converse

The primary tools used in this paper are the generalized envelope theorem by Milgrom and Segal (2002) and its generalized version with converse by Sinander (2022). We also make reference to the work of Myerson (1981), which characterizes feasible mechanisms in private value auctions with quasi-linear utility.

In a private value auction with quasi-linear utility, Myerson (1981) characterizes the feasible mechanisms based on the monotonicity of allocation, envelope formulas, individual rationality for the lowest type, and plausibility conditions. With these characterizations and a regularity assumption to ensure the monotonicity of allocation, the envelope theorem and individual rationality are equivalent to the expression $\mathbb{E}\mathbf{T}(\theta) = \int_{\theta'=0}^{\theta} \mathbb{E}\mathbf{Q}(\theta')d\theta' - \mathbb{E}\mathbf{Q}(\theta)\theta$. Consequently, the expected payment in the revenue maximization problem can be replaced by a function of the expected allocation.

In mechanism design, the choice set can be arbitrary, and the traditional envelope theorem does not hold. Milgrom and Segal (2002) provide a general version of the envelope theorem that holds for an arbitrary choice set. It is well known that outside of the quasi-linear context, the converse envelope theorem is needed to characterize feasible mechanisms. Sinander (2022) provide a generalized envelope theorem with a converse counterpart. They establish an implementability theorem using this theorem, which is a generalized version of the single crossing property with increasing allocation.

In our context, given a mechanism (Q, T), a bidder with type θ_i maximizes their distorted expected utility by choosing their report. We denote the maximum utility as $V(\theta_i) = \max_{\hat{\theta}_i \in \Theta} \mathbb{E}^{\delta}\mathbf{U}(\hat{\theta}_i, \theta_i) = f(\hat{\theta}_i, \theta_i)$. The optimal report for type θ_i is denoted as $\hat{\theta}^*(\theta_i)$.

4.3 Stochastic Ordering

In order to define an increasing allocation, it is necessary to properly order the allocation space. We utilize three stochastic orders that are particularly relevant for our analysis. These orders are integral stochastic orders, which are generated by specific sets of real-valued functions denoted as $\mathcal{F}$. For random vectors $\mathbf{X}$ and $\mathbf{Y}$, the order $\mathbf{X} \geq_{\mathcal{F}} \mathbf{Y}$ holds when $\mathbb{E}[f(\mathbf{X})] \geq \mathbb{E}[f(\mathbf{Y})]$ for every function f in the set $\mathcal{F}$ for which the expectation exists.

The increasing convex order ($\leq_{icx}$) is generated by the set of all increasing convex functions. This order measures variability, as convex functions emphasize the tail of the distribution. The convex order ($\leq_{cx}$) is generated by the set of convex functions, and it is the multivariate generalization of a mean-preserving spread. The standard order ($\leq_{st}$) is the integral stochastic order generated by the set of bounded increasing functions and serves as the multivariate generalization of first-order stochastic dominance.

We list a few properties of these stochastic orders, all of which can be found in Müller and Stoyan (2002) and Shaked and Shanthikumar (1994):

P0 If $\mathbf{X} \leq_{cx} \mathbf{Y}$, then $\mathbf{X} \leq_{icx} \mathbf{Y}$ and $\mathbf{X} \leq_{st} \mathbf{Y}$.

P1 If $\mathbf{X} \leq_{icx} \mathbf{Y}$, there exist random vectors $\mathbf{Z}$ and $\mathbf{Z}'$ such that $\mathbf{X} \leq_{st} \mathbf{Z} \leq_{cx} \mathbf{Y}$ and $\mathbf{X} \leq_{cx} \mathbf{Z}' \leq_{st} \mathbf{Y}$.

P2 For $\mathbf{X} \leq_{icx} \mathbf{Y}$, there exist random vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ (defined on the same probability space) such that $\hat{\mathbf{X}} = st\mathbf{X}$, $\hat{\mathbf{Y}} = st\mathbf{Y}$, and $E\hat{\mathbf{Y}}|\hat{\mathbf{X}} \geq E\hat{\mathbf{X}}$ almost surely. This implies that $(\hat{\mathbf{X}}, \hat{\mathbf{Y}})$ forms a submartingale.

P3 For random vectors $\mathbf{X}$ and $\mathbf{Y}$:

(a) If $\mathbf{X} \leq_{icx} \mathbf{Y}$, then $\mathbb{E}[\mathbf{X}] \leq \mathbb{E}[\mathbf{Y}]$.

(b) If $\mathbf{X} \leq_{st} \mathbf{Y}$, then $\mathbb{E}[\mathbf{X}] \leq \mathbb{E}[\mathbf{Y}]$.

(c) If $\mathbf{X} \leq_{cx} \mathbf{Y}$, then $\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{Y}]$.

In the next section, we demonstrate that the expectation operator is additive for the best-case distortion. As a result, the Myerson-type characterization of optimal mechanisms can be applied. Although the expectation is not additive for the consequential distortion, the insights gained from the best-case distortion still hold. In both cases, the optimal auction resembles a variant of the Sad-loser auction initially introduced by Riley and Samuelson (1981).

5 Optimal Auction with Best-case Binary Distortion

Assuming a regular distribution[4], which is common in the mechanism design literature, we establish two Lemmas that enable a Myerson-type characterization. The first lemma demonstrates the stability of the best case under a slight change in type, while the second lemma shows the uniqueness of the allocation and payment within each best case. Both lemmas rely on the continuity of the utility function.

Lemma 1. $\mathcal{B}(\hat{\theta}, \theta) = \mathcal{B}(\hat{\theta}, \theta + \epsilon)$ *and* $\mathbb{E}_{-i}\mathbf{U}(\theta)\delta_{\theta}(\mathbf{U}(\theta) = 0$.

Proof. Fix a report $\hat{\theta}$. By definition, $\theta_{-i} \in \mathcal{B}(\hat{\theta}, \theta)$ if for every $\theta'{-i} \notin \mathcal{B}(\hat{\theta}, \theta)$, $Q(\hat{\theta}, \theta{-i})\theta - T(\hat{\theta}, \theta_{-i}) > Q(\hat{\theta}, \theta'{-i})\theta - T(\hat{\theta}, \theta'{-i})$. This implies that for a small enough ϵ, $Q(\hat{\theta}, \theta_{-i})(\theta + \epsilon) - T(\hat{\theta}, \theta_{-i}) > Q(\hat{\theta}, \theta'{-i})(\theta + \epsilon) - T(\hat{\theta}, \theta'{-i})$. Hence, $\theta_{-i} \in \mathcal{B}(\hat{\theta}, \theta + \epsilon)$. Since the best case remains unchanged for small changes in type, we conclude that $\delta_{\theta} \equiv 0$. □

Lemma 2. *For every* $\theta_{-i}, \theta'{-i} \in \mathcal{B}(\hat{\theta}, \theta)$*,* $Q(\hat{\theta}, \theta{-i}) = Q(\hat{\theta}, \theta'{-i}) =: \bar{Q}(\hat{\theta}, \theta)$ *and* $T(\hat{\theta}, \theta{-i}) = T(\hat{\theta}, \theta'{-i}) =: \bar{T}(\hat{\theta}, \theta)$*.*

Proof. For $\theta{-i}, \theta'{-i} \in \mathcal{B}(\hat{\theta}, \theta)$, we have $Q(\hat{\theta}, \theta{-i})\theta - T(\hat{\theta}, \theta_{-i}) = Q(\hat{\theta}, \theta'{-i})\theta - T(\hat{\theta}, \theta'{-i})$. The previous Lemma shows $\mathcal{B}(\hat{\theta}, \theta) = \mathcal{B}(\hat{\theta}, \theta + \epsilon)$, which also implies that $Q(\hat{\theta}, \theta_{-i})(\theta + \epsilon) - T(\hat{\theta}, \theta_{-i}) = Q(\hat{\theta}, \theta'{-i})(\theta + \epsilon) - T(\hat{\theta}, \theta'{-i})$. Combining both

[4] $\psi(\theta) = \theta - \frac{1-F(\theta)}{\theta}$, where ψ is a monotone non-decreasing function

conditions gives $Q(\hat{\theta}, \theta_{-i}) = Q(\hat{\theta}, \theta' - i)$ and $T(\hat{\theta}, \theta - i) = T(\hat{\theta}, \theta'_{-i})$. We denote them as $\bar{Q}(\hat{\theta}, \theta), \bar{T}(\hat{\theta}, \theta)$. When $\hat{\theta} = \theta$, we denote them as $\bar{Q}(\theta), \bar{T}(\theta)$. □

With these two Lemmas, we can apply a Myerson-type characterization of optimal mechanisms to auctions with best-case binary distorted bidders. We define the distorted expected winning probability as $q(\theta_i) = (1-\delta)\mathbb{E}_{-i}\mathbf{Q}(\theta_i) + \bar{Q}(\theta_i)\delta$. Similarly, we define the distorted expected payment as $t(\theta_i) = (1-\delta)\mathbb{E}_{-i}\mathbf{T}(\theta_i) + \bar{T}(\theta_i)\delta$. Consequently, $V(\theta) = q(\theta)\theta - t(\theta)$. We can rewrite the BIC condition as $V(\theta) = V(0) + \int_0^\theta q(x)dx$. IR and profit maximization require $V(0) = 0$. Expanding $V(\theta)$ gives:

$$t(\theta_i) = q(\theta_i)\theta_i - \int_0^{\theta_i} q(x)dx. \tag{1}$$

Equation (1) summarizes BIC and IR.

The seller's objective is to maximize the expected revenue. Rearranging $t(\theta_i)$ and combining it with equation (1) gives the expected payment for type θ_i:

$$\mathbb{E}_{-i}\mathbf{T}(\theta_i) = \mathbb{E}_{-i}\mathbf{Q}(\theta_i)\theta_i - \int_{x=0}^{\theta_i} \mathbb{E}_{-i}\mathbf{Q}(x)dx + \frac{\delta}{1-\delta}\left(\bar{u}(\theta_i) - \int_{x=0}^{\theta_i} \bar{Q}(x)dx\right),$$

where $\bar{u}(\theta) = \bar{Q}(\theta)\theta - \bar{T}(\theta)$. The expected revenue from a bidder i is given by:

$$\begin{aligned}\int_{\theta_i=0}^{1} \mathbb{E}_{-i}\mathbf{T}(\theta_i)F(d\theta_i) = &\int_{\theta_i=0}^{1} \mathbb{E}_{-i}\mathbf{Q}(\theta_i)\psi(\theta_i)F(d\theta_i) \\ &+ \frac{\delta}{1-\delta}\int_{\theta_i=0}^{1} (\bar{Q}(\theta_i)\psi(\theta_i) - \bar{T}(\theta_i))F(d\theta_i),\end{aligned}$$

where $\psi(\theta_i) = \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$ is the virtual valuation when the bidder is unbiased.

Observation 1. *When the transfer to the mechanism is unbounded, the seller can achieve arbitrarily large revenue. In other words, as* $\bar{T}(\theta_i) \to -\infty$, $\int_{\theta_i=0}^{1} \mathbb{E}_{-i}\mathbf{T}(\theta_i)dF(\theta_i) \to$

∞.

This is because the mechanism designer can directly offer the bidder a chance to place a bet on another player's type. Since bidders have biased beliefs, the mechanism designer can achieve unbounded profit by offering an unfair lottery to the biased bidders, which may have nothing to do with the good itself. To focus on the incentive for obtaining the good and exclude such cases, we restrict the transfer to be positive, i.e., $\forall \theta \in \Theta^N : T(\theta) \geq 0$.

The expected revenue is decreasing in $\bar{T}(\theta_i)$. The optimal mechanism has $\bar{T}(\theta_i) = 0$. This provides a Myerson-type characterization of payment. The expected revenue per bidder is:

$$\int_{\theta_i=0}^{1} \mathbb{E}_{-i}\mathbf{T}(\theta_i)F(d\theta_i) = \int_{\theta=0}^{1} \left(\mathbb{E}_{-i}\mathbf{Q}(\theta_i) + \frac{\delta}{1-\delta}\bar{Q}(\theta_i)\right)\psi(\theta_i)F(d\theta_i). \tag{2}$$

Observation 2. *The first term,* $\int_{\theta=0}^{1} \mathbb{E}_{-i}\mathbf{Q}(\theta)\psi(\theta)F(d\theta)$*, represents the revenue when the bidders are unbiased. Therefore, bias increases the seller's revenue. The second term,* $\int_{\theta=0}^{1} \frac{\delta}{1-\delta}\bar{Q}(\theta)\psi(\theta)F(d\theta)$*, represents the surplus induced by the WT bias. It is increasing in* δ*. As a result, the more biased the bidders are, the more revenue the seller can generate.*

Observation 3. *There is no restriction on* $\mathcal{B}(\theta_i)$*.*

As the type distribution is regular, a simple solution is as follows:

$$\begin{aligned}
Q_i(\theta_i, \theta_{-i}) = 1 \qquad &\Leftrightarrow \qquad \forall j \neq i : \theta_i > \theta_j, \text{ and } \psi(\theta_i) \geq 0,\\
Q_i(\theta_i, \theta_{-i}) = \frac{1}{\bar{N}} \qquad &\Leftrightarrow \qquad \forall j \neq i : \theta_i \geq \theta_j, \text{ and } \psi(\theta_i) \geq 0,\\
Q_i(\theta_i, \theta_{-i}) = 0 \qquad &\Leftrightarrow \qquad \forall j \neq i : \theta_i < \theta_j, \text{ or } \psi(\theta_i) < 0,
\end{aligned}$$

If $\psi(\theta_i) < 0$, the bidder never wins and must have zero transfer, so the best case is $\mathcal{B}(\theta_i, \theta_i) = \Theta_{-i}$. For other cases, since there is no restriction on $\mathcal{B}(\theta_i, \theta_i)$, one can freely choose some $\mathcal{B} \subset [0, \theta_i)^{N-1}$ to make it the best case. The payment is given by equation (1) with $\bar{T}(\theta) = 0$. Here, $\bar{N}$ is the number of bidders submitting the same highest report. When there are multiple bids with positive virtual valuations, the good is allocated to the bidder with the highest valuation.

To see that the optimal mechanism has a loser-pay feature, consider a bidder i. In the best case, bidder i obtains the good for free and only has to pay in other cases. Thus, this is a sad-loser auction in the best case, which is the case where the bidder has wishful thinking and distorted beliefs.

5.1 Implementation: SPAr with Sad-loser Lottery

A Second Price Auction (SPA) with a reservation price and an additional lottery can easily implement the above mechanism. For simplicity, let's consider the case of 2 bidders with uniformly distributed valuations $\theta \sim U[0, 1]$, where types are distributed independently. In this case, the virtual valuation is $\psi(\theta_i) = 2\theta_i - 1$. We choose the best case as the other bidder reporting zero, $\mathcal{B}(\theta_i, \theta_i) = 0$.

Consider a Second-Price Auction with a reservation price of $\frac{1}{2}$, where the seller withholds the good if either bidder bids zero. In SPAr, revealing the truth is still weakly dominant. The revenue in auctions with Expected Utility Theory (EUT) bidders is given by:

$$\Pi^{SEU} = \sum_{i \in \{1,2\}, j \neq i} \left(\int_0^1 \int_0^1 (2\theta_i - 1) Q_i(\theta_i, \theta_j) d\theta_j d\theta_i \right).$$

It is worth noting that the optimal mechanism for auctions with EUT bidders can be implemented by an SPA with a reservation price of $\frac{1}{2}$, denoted as $\Pi^{\text{EUT}^*} =$

$\Pi^{\text{SPAr,EUT}}$. The SPAr, where the seller withholds the good when either bidder has a zero valuation, yields the same revenue, i.e., $\Pi^{\text{SPAr,WT}} = \Pi^{\text{SPAr,EUT}}$. Equation (2) shows that the revenue could be improved by:

$$\Pi^{WT} - \Pi^{SEU} = \sum_{i \in \{1,2\}} \left(\frac{\delta}{1-\delta} \int_0^1 (2\theta_i - 1) Q(\theta_i, 0) d\theta_i \right).$$

The optimal mechanism gives $Q(\theta_i, 0) = 1$ if $\theta_i > \frac{1}{2}$ and zero otherwise. Thus, the potential improvement is $\sum_{i \in 1,2} \frac{\delta}{4(1-\delta)}$. To extract this surplus, an extra lottery is offered to each bidder.

Recall that the seller withholds the good if either bidder has a zero valuation. The seller could offer a sad-loser lottery $L(x)$ to bidders conditional on this event. The lottery holder obtains the good for free if the other bidder bids zero (the best case) and pays x otherwise. Bidders with valuations greater than $\frac{(1-\delta)}{\delta} x$ accept the lottery. The lottery represents a bet on a null event. Whenever the lottery is accepted, the revenue is improved by x. For each bidder, the expected revenue from the lottery is:

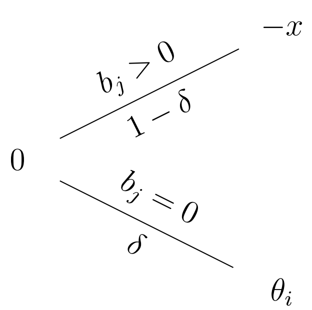
$$\Pi^{L(x)} = \int_{\frac{(1-\delta)}{\delta} x}^{1} x d\theta = x - \frac{1-\delta}{\delta} x^2.$$

The revenue is maximized when $x = \frac{\delta}{2(1-\delta)} := x^*$. The expected revenue for each bidder is $\Pi^{L(x^*)} = \frac{\delta}{4(1-\delta)}$. All possible improvements have been explored. This proves that the SPA with a reservation price of $\frac{1}{2}$ and the additional sad-loser lottery $L(x^*)$ is revenue maximizing.

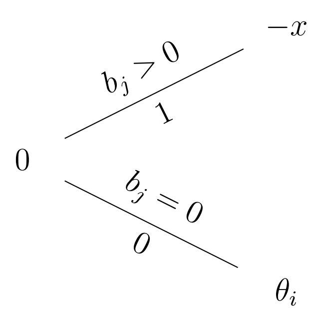
5.2 The Mechanism Behind the Sad-loser Lottery

This section explores how the sad-loser lottery improves the seller's revenue, particularly in the context of wishful thinking (WT) bidders. The lottery is profitable when bidders have biased beliefs, and the revenue is higher when the payment is conditional on loss, an underweighted event.

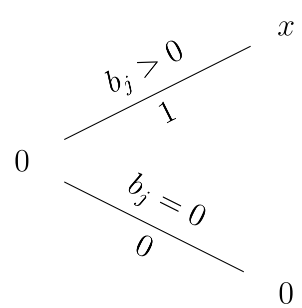
The figure below illustrates the lottery from the perspective of wishful thinking bidders, Subjective Expected Utility (SEU) maximizing bidders, and the seller. For wishful thinkers, the perceived gain of the lottery is $\delta\theta_i > 0$, and the perceived cost is $(1-\delta)x$. Bidders with valuations θ_i greater than $\frac{\delta x}{(1-\delta)}$ find the lottery profitable and accept it. EUT bidders, on the other hand, do not find the lottery profitable and reject the offer. Since the actual probability of winning is zero, it is free for the seller to provide the lottery. Anyone who accepts the lottery increases the seller's revenue by x.



(a) Perceived payoff for WT bidders



(b) Payoff for EUT bidders



(c) Payoff of the seller

Separating the case of receiving the good from paying the lottery fee always increases surplus, with the subjective expected payment remaining unchanged. SEU bidders are indifferent between paying x conditional on loss or unconditionally. However, wishful thinkers prefer paying x conditional on loss since they underweight the probability of loss. There is a perceived discount of δ on the payment when it is separated from the best case. If the lottery's payment is made unconditionally, the lottery only yields the seller an expected revenue of $\frac{\delta}{4}$ per bidder, which is suboptimal. A similar effect also arises in the winning case. Wishful thinkers have a biased perceived gain of δx from the sad-loser lottery as they overestimate the winning probability. Thus, the sad-loser feature improves revenue for mechanisms targeting wishful thinking bidders. The next section demonstrates that this intuition remains valid in the more general case of consequential distortion.

The mechanism is robust in markets with both best-case binary WT and SEU bidders. SEU bidders reject any lottery that is not in their favor, while wishful thinkers accept the lottery. We separate WT bidders from EUT bidders, even if the bidders themselves fail to recognize their bias. However, to offer the optimal sad-loser lottery, the seller must know the parameter δ.

6 Consequential Wishful Thinking

This section examines the auction problem with wishful thinking (WT) modeled by the more general consequential distortion. We demonstrate that under a mild condition on the shape of the distortion function v, the optimal mechanism is a variant of the loser-pay auction, and the bias increases the seller's revenue.

We require the distortion function to satisfy the following condition:

$$\forall u \leq 1 : v''(u) \leq 2v'(u) + uv''(u). \tag{3}$$

This condition ensures that the agent's preference is monotonic with respect to first-order stochastic dominance (FOSD).[5] It also guarantees a convex distorted expected utility.

Lemma 3. *The distorted preference satisfies FOSD when*

$$\forall u \leq 1 : v'(u) \leq v(u) + uv'(u). \tag{4}$$

The condition (4) is always satisfied when $u = 1$ since $v > 0$. By the Fundamental Theorem of Calculus, (3) can be seen as a smooth version of (4).

Before delving into the problem of the optimal auction with wishful bidders, let us review some basic properties of consequential distortion. Based on these properties, we will define the maximum spread auction, which intuitively should be the optimal auction. Finally, we will prove that it is indeed the case under certain assumptions.

First, we observe that wishful thinkers prefer a spread.

Lemma 4. *Given a set of acts (reports) with the same subjective (unbiased) expected payoff, wishful thinkers prefer the one with a greater spread.*

Proof. Define the function $\phi(u) = uv(u)$. Then, the distorted expectation for a random variable (act) $\mathbf{u}$ can be written as $\frac{\mathbb{E}\phi(\mathbf{u})}{\mathbb{E}v(\mathbf{u})}$. Approximating the value of the distorted expected value around the mean gives:

$$\mathbb{E}^{\delta}\mathbf{u} = \frac{\mathbb{E}\phi(\mathbf{u})}{\mathbb{E}v(\mathbf{u})} \approx \frac{\phi(\mathbb{E}\mathbf{u}) + \phi''(\mathbb{E}\mathbf{u})Var(\mathbf{u})}{v(\mathbb{E}\mathbf{u}) + v''(\mathbb{E}\mathbf{u})Var(\mathbf{u})}.$$

Since (3) ensures $\phi'' > v''$, for two random variables $\mathbf{u}$ and $\mathbf{u}'$ with $\mathbf{u}'$ being the mean-preserved spread of $\mathbf{u}$, $Var(\mathbf{u}') > Var(\mathbf{u})$. Thus, $\mathbb{E}^{\delta}\mathbf{u}' > \mathbb{E}^{\delta}\mathbf{u}$. □

The agent's beliefs are increasingly biased toward events with good outcomes.

[5] For random variables X and Y, if X first-order stochastically dominates Y ($X >_{st} Y$), then the agent prefers X over Y ($X \succ Y$).

Whenever there is a spread between good and bad events, while keeping the expected outcome unchanged, the agent will believe that she is more likely to win and likely to pay less. Since the maximum utility for a type θ agent is θ, and the minimum payoff is $-\bar{t}$, a maximum exists for every distribution of states (others' types).

Figure 1 provides an intuitive explanation. The black curve represents the state-dependent random utility $u(\theta_{-i})$, and the red line depicts the mean-preserved deviation $u'(\theta_{-i})$ of $u(\theta_{-i})$. Suppose that for every $\theta^+_{-i} \in \Theta^+_{-i}$ and $\theta^-_{-i} \in \Theta^-_{-i}$, $u(\theta^+_{-i}) > u(\theta^-_{-i})$. By increasing $u(\theta_{-i})$ in Θ^+_{-i} by a fixed amount Δ while preserving the mean, we can reduce $u(\theta_{-i})$ in Θ^-_{-i} by $\Delta\frac{G(\Theta^-_{-i})}{G(\Theta^+_{-i})}$. Since $u'(\theta_{-i})$ in Θ^+_{-i} is increased, while $u'(\theta_{-i})$ in Θ^-_{-i} is decreased, the distorted agent further increases the distorted probability assessment on Θ^+_{-i} and decreases those on Θ^-_{-i}. As a result, the distorted utility for u' is greater than u. By continuing to increase the distorted utility through the selection of two regions and increasing the spread, two possible cases emerge. The support of the utility function is either $\theta, 0, -\bar{t}$ or $\theta, \theta - \bar{t}, -\bar{t}$. Figures 2 and 3 illustrate these two cases.

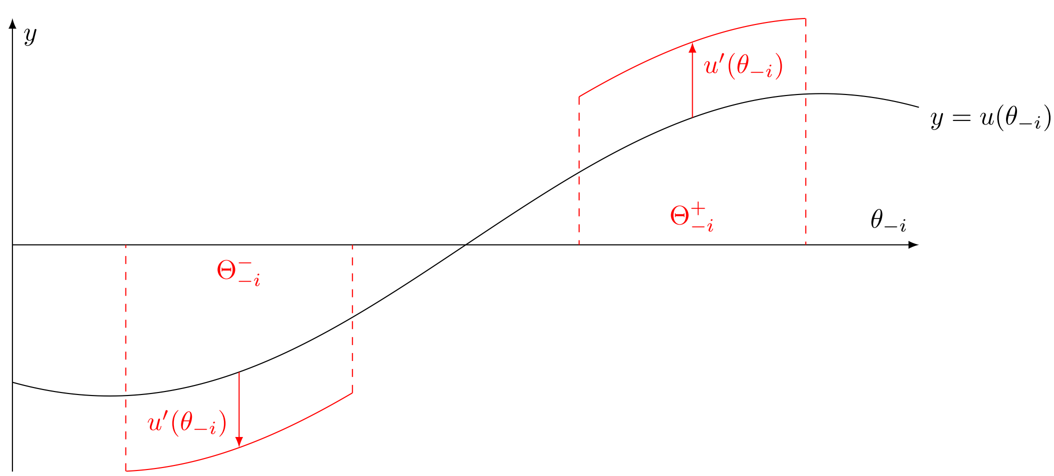


Figure 1: Mean-preserved Spread of u that increases the distorted expectation.

This provides intuition as to why the loser-pay auction is optimal in the previous section. Naturally, it offers the maximum spread to the agent. The seller has no preference for the distribution of the winning probability and the payment for any report. At the same time, the bidder prefers the one that offers the greatest spread to the generated random utility. We consider the case where some report does not provide the maximum spread to the bidder as a potential for Pareto improvement. The seller could extract the surplus from the improvement with an appropriate mechanism. Thus, we formally define the maximum spread report and the maximum spread auction and then show their optimality under certain additional assumptions.

Definition 1 (Maximum spread report)**.** Given a mechanism $\mathbf{Q}, \mathbf{T}$, a report θ is a maximum spread report when the random utility it generates for the corresponding type, $\mathbf{U}(\theta) = \theta\mathbf{Q}(\theta) - \mathbf{T}(\theta)$, is $\geq_{cx}$-maximum among the mechanisms $\mathbf{Q}', \mathbf{T}'$ that give the same expected winning probability and payment for θ, i.e., $\mathbb{E}_{-i}\mathbf{Q}'(\theta) = \mathbb{E}_{-i}\mathbf{Q}(\theta)$ and $\mathbb{E}_{-i}\mathbf{T}'(\theta) = \mathbb{E}_{-i}\mathbf{T}(\theta)$.

The maximum is well-defined as the support is bounded.

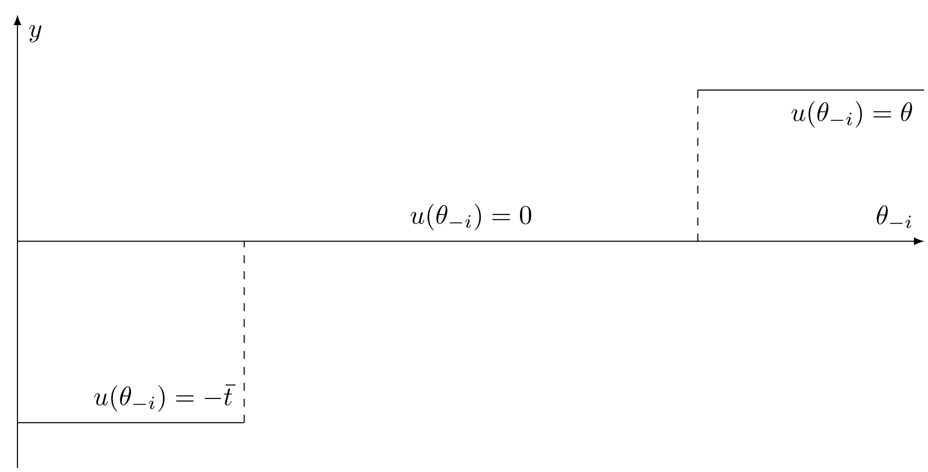


Figure 2: Case 1: Support of the utility function is $\theta, 0, -\bar{t}$.

Definition 2 (Maximum spread auction)**.** A mechanism is called a maximum spread auction when all reports are maximum spread reports.

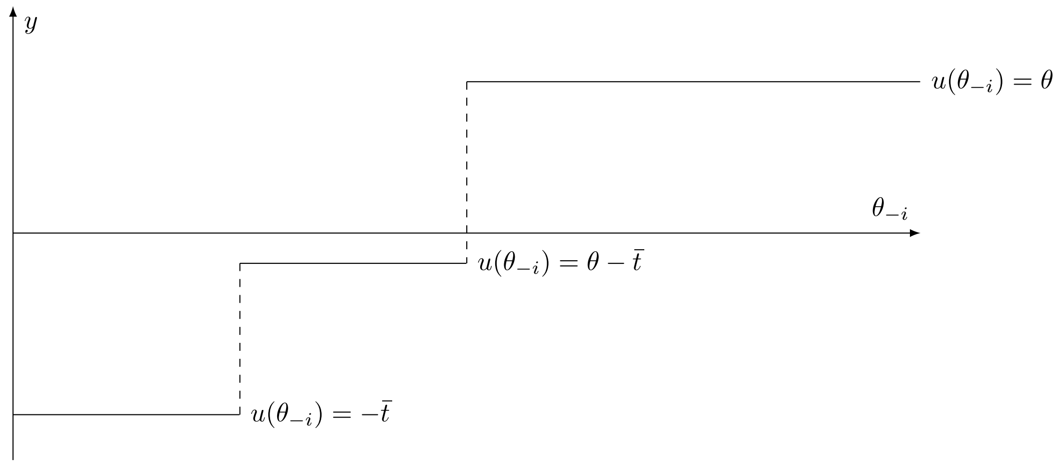


Figure 3: Case 2: Support of the utility function is $\theta, \theta - \bar{t}, -\bar{t}$.

This means that there are functions $\alpha, \beta : \Theta \to [0, 1]$ such that

$$G[\theta_{-i} : Q(\theta, \theta_{-i}) = 1 \text{ and } T(\theta, \theta_{-i}) = 0] = 1 - \beta(\theta),$$

$$G[\theta_{-i} : Q(\theta, \theta_{-i}) = 0 \text{ and } T(\theta, \theta_{-i}) = \bar{t}] = 1 - \alpha(\theta),$$

$$G[\theta_{-i} : Q(\theta, \theta_{-i}) = 1 \text{ and } T(\theta, \theta_{-i}) = \bar{t}] = (\alpha(\theta) + \beta(\theta) - 1)^{+},$$

$$G[\theta_{-i} : Q(\theta, \theta_{-i}) = 0 \text{ and } T(\theta, \theta_{-i}) = 0] = (1 - \alpha(\theta) - \beta(\theta))^{+},$$

where $(z)^{+} = \max z, 0$. Here, $\alpha(\theta)$ is the probability of getting the goods, and $\beta(\theta)$ is the probability of paying the whole budget.

A necessary condition is that for every $\theta_i \in \Theta, \theta_{-i} \in \Theta_{-i}$:

$$Q(\theta_i, \theta_{-i}) \in 0, 1 \; T(\theta_i, \theta_{-i}) \qquad \in 0, \bar{t}.$$

The maximum spread auction is a variation of the sad-loser auction, which, in turn, is a variation of an all-pay auction. Our main theorem shows that it is optimal when the budget is large, and the expected winning probability schedule is increasing.

Theorem 1 (Main Theorem). *When the budget is large and the expected winning schedule, $\mathbb{E}_{-i}\mathbf{Q}$, is an increasing function, the optimal mechanism is of maximum spread.*

An immediate corollary of the theorem is the following.

Corollary 1 (Main Theorem (Alternative)). *When the budget is large, the optimal efficient mechanism is of maximum spread.*

To prove the main theorem, we will show that when the allocation schedule increases, individual rationality (IR) is met for the lowest type, and the envelope formula is satisfied and plausible, then the mechanism is feasible. We will mainly demonstrate that IR for the lowest type implies IR for all types, and we define the allocation space and order in such a way that the Spence-Mirrlees condition implies that when the envelope formula holds, the Bayesian incentive compatibility (BIC) condition also holds.

Proposition 1. *When condition (3) holds, for any mechanism that satisfies BIC, $V(0) \geq 0 \Rightarrow \forall \theta \in \Theta : V(\theta) \geq 0$ (IR is satisfied).*

Proof. Note that the same report from a higher type first-order stochastically dominates the same report from lower types. By Lemma 2, (3) implies that for every type θ_i, $\mathbb{E}^{\delta}_{-i}\mathbf{U}(0, \theta_i) \geq \mathbb{E}^{\delta}_{-i}\mathbf{U}(0, 0)$. BIC requires $\mathbb{E}_{-i}\mathbf{U}(\theta_i, \theta_i) \geq \mathbb{E}_{-i}\mathbf{U}(0, \theta_i)$. As V is increasing, $V(0) \geq 0 \Rightarrow \forall \theta \in \Theta : V(\theta) \geq 0$. □

Proposition 1 replicates the result in Myerson (1981) that the IR constraint only restricts the initial value of the envelope formula. Next, we construct the implementability theorem as in (Sinander, 2022) in our context. With the implementability theorem, any increasing allocation is implementable: there is a payment schedule such that the mechanism is BIC. The payment schedule and the allocation must jointly satisfy the envelope formula.

Next, we give an implementability theorem. The implementability theorem isolates payment, which the seller cares about, from allocations, for which the bidder has a preference. In our context, the bidders have a preference over the distribution of payment. Thus, we extend the allocation space to include the distribution of both payment and winning probability. For a mechanism $\mathbf{Q}, \mathbf{T}$, we separate the expected payment from its distribution by defining functions $\tau, \mathbf{P}$ from the payment schedule $\mathbf{T}$. $\tau : \Theta \to [0, k]$ is the expected payment schedule, $\tau = E\mathbf{T}$. $\mathbf{P} : \Theta \to \mathbb{R}^+$ is the proportion of the expected payment schedule, $\mathbf{P} = \frac{\mathbf{T}}{\tau}$. When $\tau = 0$, which implies $\mathbf{T} = 0$, we define $\mathbf{P} = 0$. The allocation schedule of a mechanism is denoted as $\mathbf{Y} = (\mathbf{Q}, \mathbf{P})$. Thus, a mechanism can be written as $\mathbf{M} = (\mathbf{Q}, \mathbf{T}) = (\mathbf{Q}, \mathbf{P}, \tau) = (\mathbf{Y}, \tau)$. The allocation space and payment space are defined as $\mathcal{Y} = \mathcal{Q} \times \mathcal{P}$ and $\mathcal{P}$, respectively.

We order the allocation space $\mathcal{Y}$ by the following order $\geq_{\mathcal{Y}}$. For $\mathbf{y}, \mathbf{y}' \in \mathcal{Y}$, with $\mathbf{y} = (\mathbf{q}, \mathbf{p})$ and $\mathbf{y}' = (\mathbf{q}', \mathbf{p}')$, we have $\mathbf{y}' \geq_{\mathcal{Y}} \mathbf{y}$ if $(\mathbf{p}', -\mathbf{t}') \geq_{icx} (\mathbf{p}, -\mathbf{t})$.

The implementability theorem of (Sinander, 2022) suggests that if $\mathcal{Y}$ is regular and the payoff function f satisfies the outer Spence-Mirrlees condition, then any increasing allocation is implementable.

[6] [7] and f satisfies the outer Spence-Mirrlees condition [8], then any increasing

[6]The outcome space $\mathcal{Y}$ is *regular* iff it is order-dense-in-itself, countably chain-complete and chain-separable.

[7]For payoff function f that maps allocation, y, payment, p, and type, t, to payoff, $f(y, p, t)$. f is *regular* iff (i) the type derivative f_3 exists and is bounded, and $f_3(y, \cdot, t)$ is continuous for each $y \in \mathcal{Y}$ and $t \in [0, 1]$, and (ii) for every chain $\mathcal{C} \subseteq \mathcal{Y}$, f is jointly continuous on $\mathcal{C} \times \mathbb{R} \times [0, 1]$ when $\mathcal{C}$ has the relative topology inherited from the order topology on $\mathcal{Y}$.

[8]f satisfies the (strict) *outer Spence-Mirrlees condition* iff for any increasing $\mathbf{Y} : [0, 1] \to \mathcal{Y}$, any $\tau : [0, 1] \to \mathbb{R}$ and any $r < t$ in $(0, 1)$,

$$n \mapsto \frac{\bar{d}}{\bar{d}m} \int_r^t f(\mathbf{Y}(s+m), \tau(s+m), s+n) ds \bigg|_{m=0}$$

allocation is implementable[9].

The Spence-Mirrlees condition is a single crossing property on the payoff function that interprets as the higher type is more willing to pay for an increase in allocation. In the appendix, we show that if we focus on a restricted set of mechanisms, the outer Spence-Mirrlees condition holds.

The restricted set of the mechanism is such that for every $\theta' > \theta$, $\mathbf{Y}(\theta') = (\mathbf{q}', \mathbf{p}'), \tau(\theta') = t', \mathbf{Y}(\theta) = (\mathbf{q}, \mathbf{p}), \tau(\theta) = t$, $\hat{\mathbf{U}}(\theta) = \mathbf{q}\theta - t\mathbf{p}$:

$$\mathbb{E}_{-i}\{\phi'(\hat{\mathbf{U}}(\theta))(\mathbf{q}' - \mathbf{q})\} \geq \mathbb{E}_{-i}\{[\phi'' - v''](\hat{\mathbf{U}}(\theta))\mathbf{q}\mathbf{p}\}(t' - t) \text{ for every } \theta \in [0, 1]. \quad (5)$$

To see why this restriction is needed, we could focus on the right-hand side of the inequality. When $\mathbf{q}$ and $\mathbf{p}$ have separated support, the right-hand side is equal to zero. (5) holds by the property O2. When the support for payment and winning is mixed, there are two effects of increasing type on the willingness to pay. First, the higher types have a greater valuation for winning, thus giving a more distorted probability assessment and expected value. On the other hand, when the support for payment and winning are overlapped, higher types have greater utility for the region of payment, thus increasing the distorted expected payment. As a result, the aggregate effect is ambiguous. The second effect reduces when the payment and winning are less aligned, $\mathbb{E}_{-i}\mathbf{q}\mathbf{p}$ reduces. We say the mechanism has sufficiently separated regions of winning and payment if (5) holds. This gives the following implementability theorem.

Proposition 2. *Any mechanism with sufficiently separated regions of winning and payment with increasing allocation is implementable.*

---

is (strictly) single-crossing, where $\bar{d}/\bar{d}m$ denotes the upper derivative.

[9]An allocation $\mathbf{Y} : [0, 1] \to \mathcal{Y}$ is *implementable* iff there is a payment schedule $\tau : [0, 1] \to \mathbb{R}$ such that $(\mathbf{Y}, \tau)$ is incentive-compatible. An increasing allocation is one that provides higher types with larger outcomes (in the partial order on $\mathcal{Y}$).

With the implementability theorem in hand, we could fix any increasing allocation and look for the payment schedule using the envelope formula. When 5 holds, the result mechanism must be BIC.

Next, we solve the problem of determining which mechanism with an increasing allocation is optimal. We seek a profitable deviation for feasible mechanisms that are not of maximum spread. Since the agent prefers spread, the target deviation is to increase the spread for the mechanism. We think of this as a Pareto improvement and then seek a mechanism to extract the generated surplus.

First, we show that for any increasing allocation with some non-maximum spread reports, we can replace those reports with reports of greater spread. The resulting allocation is still increasing, so increasing the spread does not change the monotonicity of the allocation.

Lemma 5. *For a mechanism with an increasing allocation that has reports that are not of maximum spread, there is another mechanism with an increasing allocation but with increased spread for the non-maximum spread reports in the original mechanism.*

Proof. Suppose the increasing mechanism is $\mathbf{y}_1 \leq_{icx} \mathbf{y}_2 \leq_{icx} \mathbf{y}_3$ with $\mathbf{y}_2$ not of maximum spread. Then, by property O1, there exists $\mathbf{z} \neq_{st} \mathbf{y}_2$ such that $\mathbf{y}_1 \leq_{icx} \mathbf{y}_2 \leq_{cx} \mathbf{z} \leq_{st} \mathbf{y}_3$. As $\mathbf{y}_1 \leq_{icx} \mathbf{z} \leq_{st} \mathbf{y}_3$, there exists $\mathbf{z}'$ such that $\mathbf{y}_1 \leq_{cx} \mathbf{z}' \leq_{st} \mathbf{z} \leq_{st} \mathbf{y}_3$. Note that there is an interval between $\mathbf{z}$ and $\mathbf{z}'$ in the *st* order where we can replace $\mathbf{y}_2$ with some $\mathbf{y}_2'$ between $\mathbf{z}$ and $\mathbf{z}'$ in the *st* order. We can choose the one with a higher spread, $\mathbf{y}_2' \geq_{cx} \mathbf{y}_2$. □

Next, we show that when we disregard the monotonicity constraints, the seller would always want to offer a mechanism with maximum spread.

Lemma 6. *If* $\mathbf{U}(\theta|Q', T') \geq_{cx} \mathbf{U}(\theta|Q, T)$*, then* $f_\theta(\theta|Q', T') > f_\theta(\theta|Q, T)$*, and* $f_\theta(\theta)$ *is an increasing function.*

The lemma shows that when we replace a non-maximum spread report with one that has more spread, both the indirect utility and its slope increase.

Lemma 7. $V(\theta|\mathbf{Q},\mathbf{T})$ *and* $f_\theta(\theta|\mathbf{Q},\mathbf{T})$ *are increasing in the probability of winning schedule* $\mathbf{Q}(\theta)$ *and decreasing in the payment schedule* $\mathbf{T}(\theta)$*, both ordered by first-order stochastic dominance (the st order). For changes in* $\mathbf{Q}$ *or* $\mathbf{T}$ *that have the same impact on the resulting random utility,* $\mathbf{U}(\theta|\mathbf{Q}',\mathbf{T}) = \mathbf{U}(\theta|\mathbf{Q},\mathbf{T}')$*, the relative impact on* V *and* f_θ *is different.*

For a fixed θ, let $\mathbf{q} = \mathbf{Q}(\theta)$ and $\mathbf{t} = \mathbf{T}(\theta)$. $V(\mathbf{q},\mathbf{t}) = V(\theta)$ and $f_\theta(\mathbf{q},\mathbf{t}) = f_\theta(\theta)$. For a functional F that takes functions a and b as inputs, let η be the derivative of F with respect to a, i.e., $F_a^\eta(a,b) = \lim_{\epsilon\to 0}\frac{F(a+\epsilon\eta,b)-F(a,b)}{\epsilon}$. For every $\eta^q \leq 0$ and $\eta^t \geq 0$:

$$
\begin{aligned}
V_q^{\eta^q} &\leq 0, & f^{\eta^q}\theta, q(\mathbf{q},\mathbf{t}) &\leq 0,\\
V_t^{\eta^t} &\leq 0, & f^{\eta^t}\theta, t(\mathbf{q},\mathbf{t}) &\leq 0.\\
\text{For every } \eta : \Theta_{-i} \to \mathbb{R}^+, & & \frac{V_q^{\theta\eta}}{f_{\theta,q}^{\theta\eta}}(\mathbf{q},\mathbf{t}) &\neq \frac{V_t^{-\eta}}{f_{\theta,t}^{-\eta}}(\mathbf{q},\mathbf{t})
\end{aligned}
$$

Lemma 7 suggests that when we replace a non-maximum spread report with one that has more spread, the value function increases. Lemma 8 shows that we can reduce it back to the original value function by decreasing the probability of winning or increasing the payment in the *st* order. In the first case, we have some free winning probability to assign. We can assign it to the report that has the greatest payment, giving more revenue to the seller as the report with the greatest payment becomes more attractive. In the second case, it is obvious that the deviation is profitable. In sum, Lemmas 7 and 8 show that when both mechanisms are feasible, the optimal mechanism is the one with maximum spread.

We return to the implementability condition that requires an increasing allocation and (5) to hold. Given that the maximum spread auction is feasible when the budget is large and the expected winning probability schedule is increasing, (5) naturally holds. Also, note that the maximum spread auction with an increasing expected winning schedule implies an increasing allocation, which proves our main theorem.

In conclusion, the main theorem states that when the budget is large and the expected winning schedule is an increasing function, the optimal mechanism is of maximum spread. The proof is feasible because if a mechanism satisfies the envelope formula, and we have another mechanism that gives the same values of $V(\theta)$, $f_\theta(\theta)_{\theta\in\Theta}$, the alternative mechanism also satisfies the same envelope formula. We find a process in which the revenue increases while keeping the envelope formula unchanged, and we show that the optimal mechanisms from the previous process are implementable.

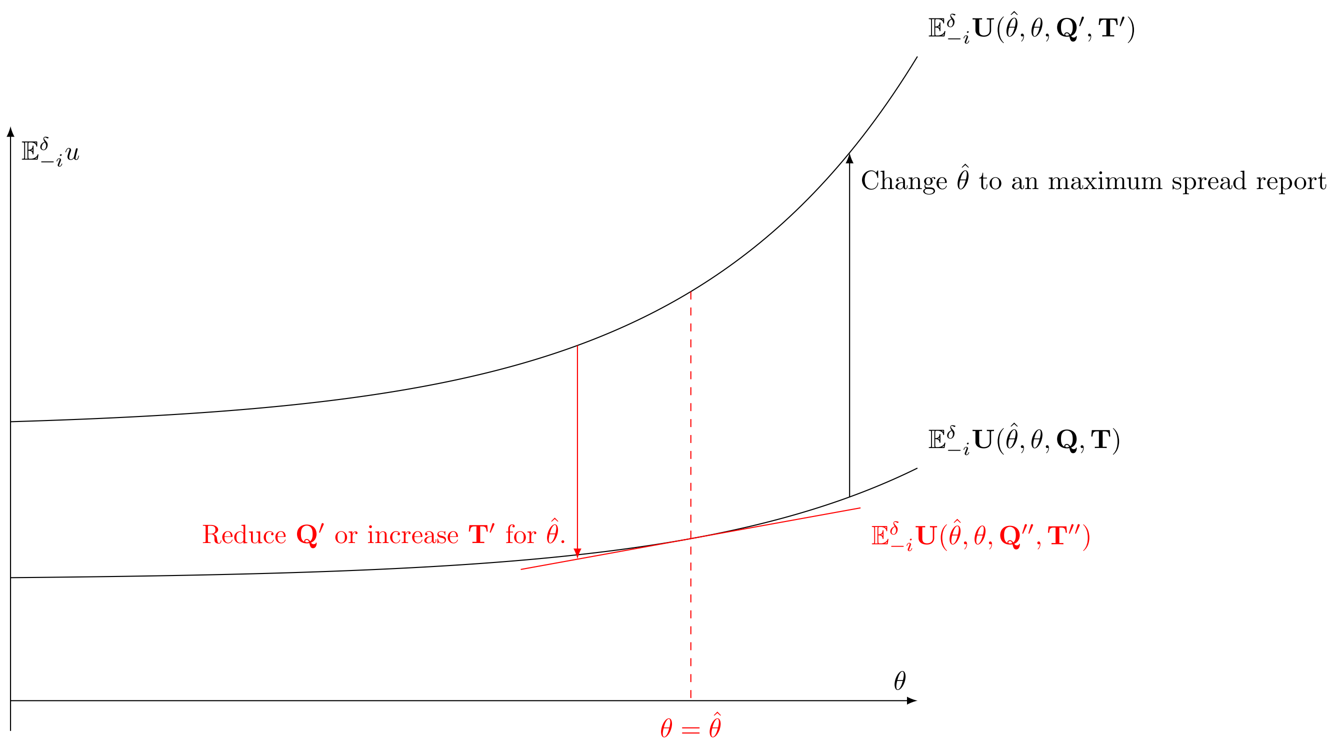


Figure 4: The non-maximum spread report is not optimal.

Figure 4 provides a conceptual understanding of the proof. The figure plots the expected utility derived from a fixed report $\hat{\theta} = \theta$ as a function of the true type Θ. Given a mechanism $(\mathbf{Q}, \mathbf{T})$ which yields a non-maximum spread report θ, the distorted expected utility (and similarly, the slope) is augmented by altering the report θ to a maximum spread one in $(\mathbf{Q}', \mathbf{T}')$. The seller can then seize the surplus by suitably decreasing allocation and increasing payment such that the new mechanism $(\mathbf{Q}'', \mathbf{T}'')$ delivers the same expected utility function for report θ as found in the original mechanism $(\mathbf{Q}, \mathbf{T})$. Consequently, the initial envelope formula applies, and revenue surges, demonstrating that when the maximum spread auction is viable, it is the optimal choice. The proof is finalized by illustrating that if the budget is ample and the expected winning probability schedule is ascending, the maximum spread auction is feasible.

6.1 Comparative Statics with Respect to the Level of Distortion

In this section, we examine how the revenue is affected by the level of distortion. First, we compare the revenue between auctions with unbiased bidders and auctions with wishful thinking (WT) bidders as a baseline. Recall that WT bidders make accurate assessments of the probability of purely random events generated by the mechanism but have distorted beliefs about others' types. The optimal mechanism for selling to unbiased bidders can be implemented as a constant mechanism for wishful thinkers by absorbing the randomness from profiles and randomizing the allocation and payment itself. As a result, the wishful thinker becomes unbiased. Therefore, the revenue from auctions with WT bidders is weakly greater than the revenue from auctions with unbiased bidders. This result is summarized in the following proposition.

Proposition 3. *The revenue of the seller in auctions with WT bidders is weakly greater than that with unbiased bidders.*

Next, we investigate how the revenue changes with the level of distortion. To do this, we define the notion of "more distorted."

Definition 3 (The distortion function v_2 is more wishful than v_1)**.** v_2 is more wishful than v_1 if, for any two random variables $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X}$ first-order stochastically dominates $\mathbf{Y}$,

$$\mathbb{E}^{\delta(v_2)}\mathbf{X} > \mathbb{E}^{\delta(v_1)}\mathbf{X} \tag{6}$$

$$\mathbb{E}^{\delta(v_2)}\mathbf{X} - \mathbb{E}^{\delta(v_2)}\mathbf{Y} > \mathbb{E}^{\delta(v_1)}\mathbf{X} - \mathbb{E}^{\delta(v_1)}\mathbf{Y} \tag{7}$$

These two conditions ensure that when v_2 is more distorted than v_1, the increase in expected utility under distortion of v_2 is greater than that under v_1 (condition (6)). Moreover, the increase in the spread of expected utility is also greater under v_2 than under v_1 (condition (7)).

To understand how the revenue is affected by the level of distortion, consider that if v_2 is more distorted than v_1, the distorted expected utility f_θ under v_2 is greater than that under v_1 due to condition (7). As a result, the seller can mimic any mechanism for v_1 under the context of v_2 by reducing allocation and increasing payments. This leads to the following proposition.

Proposition 4. *The revenue is increasing in the level of wishfulness.*

The proof is as follows: Suppose (Q, T) is the optimal mechanism for less distorted bidders with distortion function v_1. Given the utility distribution $V(\theta), f_\theta(\theta)\theta \in \Theta$ that it generates, the same (Q, T) will yield a greater utility distribution $V(\theta)', f\theta(\theta)'\theta \in \Theta$

with more distorted bidders using v_2. By Lemma 6, we can reduce it back to the original utility distribution $V(\theta), f\theta(\theta)_{\theta\in\Theta}$ by increasing T and reducing Q. With this change, the new mechanism generates more profit and remains feasible.

As a corollary of the previous propositions, we can simplify the seller's optimization problem under the assumption of a large budget and an increasing expected winning probability schedule, denoted as $\mathbb{E}_{-i}\mathbf{Q}$:

Corollary 2. *Under the assumptions of a large budget and an increasing expected winning probability schedule, the seller's optimization problem can be simplified as follows:*

$$\begin{aligned}
\max_{a,b,c,d\in\Theta^{[0,1]}} \int_{\Theta} b(\theta) + c(\theta)F(d\theta)\bar{t} \\
s.t. \int_{\theta'=0}^{\theta} f_\theta(\theta')d\theta' =& \frac{\sum_{p\in x} p(\theta_i)v_p(\theta_i)u_p(\theta_i)}{\sum_{p\in x} p(\theta_i)v_p(\theta_i)} \text{ for every } \theta\in\Theta, \\
f_\theta(\theta) =& \frac{\left(\sum_{p\in w} p(\theta)(v_p(\theta) + u_p(\theta)v'p(\theta))\right)\left(\sum p\in xp(\theta)v_p(\theta)\right)}{\left(\sum_{p\in x} p(\theta)v_p(\theta)\right)^2} \\
&- \frac{\left(\sum_{p\in w} p(\theta)v'p(\theta)\right)\left(\sum p\in xp(\theta)u_p(\theta)v_p(\theta)\right)}{\left(\sum_{p\in x} p(\theta)v_p(\theta)\right)^2}, \\
a+b+c+d =& 1, \\
\int_{\Theta} a(\theta) + c(\theta)F(d\theta) =& \frac{1}{N}, \text{ and } d(0) = 1,
\end{aligned}$$

where a, b, c, d represent the probabilities of all possible cases, and $x = a, b, c, d$ and $w = a, c$ represent the winning cases. The utility and distortion functions for each case are denoted as $u_p(\theta)$ and $v_p(\theta)$, respectively.

With consequential distortion, mechanisms that yield the same expected allocation and payment for every type can result in different revenues. Therefore, the seller needs to consider the profile-dependent utility distribution for each report. With the above propositions, the seller can focus on maximum spread auctions, and the problem becomes assigning winning and losing probabilities for each type. Instead of choosing (Q, T) functions that map Θ^N to $[0, 1]$ and $[0, \bar{t}]$, the seller can focus on choosing (a, b, c, d) functions that map types in Θ to probabilities in $[0, 1]$. This simplifies the dimensionality of the problem. Another possibility is to use the (α, β) functions as defined above, which further reduces the dimensionality of the problem but introduces some non-smoothness due to the use of the $(\cdot)^+$ operator.

7 Discussion

Wishful thinking (WT) leads bidders in standard auctions to overestimate their chances of winning, prompting underbidding as they underestimate competitors' valuations. This miscalculation of payment expectancy reduces the potential surplus, lowering the seller's revenue. However, the implementation of sad-loser auctions can reverse this trend, enabling the seller to profit from bidder bias. These auctions further allow the mechanism designer to manipulate the level of distortion by adjusting the outcome distribution. Thus, the more biased the bidders, the greater the seller's profit.

Sad-loser auctions, while not common in goods sales, appear frequently in contracts and market structures like the 'loser pays attorney fees' clauses and R&D processes. This type of system, where 'the winner gets all,' can incentivize WT individuals to participate. Whether this is beneficial depends on the designer's objectives. Given the correlation between optimism and traits like creativity (Rego et al., 2012), risk-taking (Anderson and Galinsky, 2006), and procrastination (Sigall et al., 2000), the implications vary. 'Loser-pays' contracts may be unsuitable for hiring or loans if WT individuals are prone to risky decisions, but may foster creativity in R&D

contests.

Our findings indicate that WT individuals underbid in private value auctions. However, the literature suggests that bidders in common value auctions may suffer from the winner's curse, bidding above true value due to underestimated correlations between others' actions and information (Easley and Ghosh, 2015). Future research could investigate how WT individuals behave in common value auctions when subject to similar biases.

References

Anderson, C. and A. D. Galinsky (2006): "Power, optimism, and risk-taking," *European journal of social psychology*, 36, 511–536.

Cohen, C. and A. Sela (2005): "Manipulations in contests," *Economics Letters*, 86, 135–139.

Cohen, N. S. (1992): "The effect of singing instruction on the speech production of neurologically impaired persons," *Journal of Music therapy*, 29, 87–102.

Cox, J. C., V. L. Smith, and J. M. Walker (1988): "Theory and individual behavior of first-price auctions," *Journal of Risk and uncertainty*, 1, 61–99.

Easley, D. and A. Ghosh (2015): "Behavioral mechanism design: Optimal crowdsourcing contracts and prospect theory," in *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, 679–696.

Engelbrecht-Wiggans, R. (1989): "The effect of regret on optimal bidding in auctions," *Management Science*, 35, 685–692.

Engelmann, J. B., M. Lebreton, N. A. Salem-Garcia, P. Schwardmann, and J. J. van der Weele (2023): "Anticipatory Anxiety and Wishful Thinking," .

Gagnon-Bartsch, T., M. Pagnozzi, and A. Rosato (2021): "Projection of Private Values in Auctions," *American Economic Review*, 111, 3256–3298.

Gilboa, I. and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of mathematical economics*, 18, 141–153.

Granberg, D. and E. Brent (1983): "When prophecy bends: The preference–expectation link in US presidential elections, 1952–1980." *Journal of Personality and Social Psychology*, 45, 477.

Hogarth, R. M. (1987): *Judgement and choice: The psychology of decision*, John Wiley & Sons.

Kagel, J. H. and D. Levin (2008): "Auctions: A survey of experimental research, 1995-2008," *Handbook of experimental economics*, 2.

Kovach, M. (2020): "Twisting the truth: Foundations of wishful thinking," *Theoretical Economics*, 15, 989–1022.

Krizan, Z. and P. D. Windschitl (2007): "The influence of outcome desirability on optimism." *Psychological bulletin*, 133, 95.

Matros, A. (2012): "Sad-loser contests," *Journal of Mathematical Economics*, 48, 155–162.

Mayraz, G. (2011): "Wishful thinking," *Available at SSRN 1955644*.

Milgrom, P. and I. Segal (2002): "Envelope theorems for arbitrary choice sets," *Econometrica*, 70, 583–601.

Minchuk, Y. (2018): "Effect of reimbursement on all-pay auction," *Economics Letters*, 172, 28–30.

Morgan, J., K. Steiglitz, and G. Reis (2003): "The spite motive and equilibrium behavior in auctions," *Contributions to Economic Analysis & Policy*, 2, 1102.

Müller, A. and D. Stoyan (2002): *Comparison Methods for Stochastic Models* *and Risks.*

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of operations research*, 6, 58–73.

Neugebauer, T. and R. Selten (2006): "Individual behavior of first-price auctions: The importance of information feedback in computerized experimental markets," *Games and Economic Behavior*, 54, 183–204.

Nikolova, E., E. Pountourakis, and G. Yang (2018): "Optimal mechanism design with risk-loving agents," in *Web and Internet Economics: 14th International Conference, WINE 2018, Oxford, UK, December 15–17, 2018, Proceedings 14*, Springer, 375–392.

Rego, A., F. Sousa, C. Marques, and M. P. e. Cunha (2012): "Optimism predicting employees' creativity: The mediating role of positive affect and the positivity ratio," *European Journal of Work and Organizational Psychology*, 21, 244–270.

Riley, J. G. and W. F. Samuelson (1981): "Optimal auctions," *The American Economic Review*, 71, 381–392.

Rosato, A. (2023): "Loss aversion in sequential auctions," *Theoretical Economics*, 18, 561–â596.

Shaked, M. and J. G. Shanthikumar (1994): *Stochastic orders and their applications*, Academic press.

Sigall, H., A. Kruglanski, and J. Fyock (2000): "Wishful thinking and procrastination," *Journal of social behavior and personality*, 15, 283–296.

Sinander, L. (2022): "The converse envelope theorem," *Econometrica*, 90, 2795–2819.

Tversky, A. and D. Kahneman (1992): "Advances in prospect theory: Cumulative representation of uncertainty," *Journal of Risk and uncertainty*, 5, 297–323.

A Appendix

Here we first provide 2 simple proof A.1 and A.2. that give intuition on how things work. In A.3. we give a generalized proof that is the work horse for all other proofs, which the generalized proof of A.1 and A.2 also nested in it.

A.1 Condition for FOSD (make sure $V(0) \geq 0 \Rightarrow V(\theta) \geq 0$)

Suppose we have a payoff function, $x(i)$, depends on some states $i \in 1, ..., n$, with probability p_i. For notaton ease we denote $x(i)$ as x_i, and x refers to $\{x_i\}_{i=1}^n$. Now we derive the condition under which the distortion function preserves the FOSD property of the subjective utility. Suppose we have some positive deviation function depends on states $\epsilon(i) \geq 0$ for each i and define a new lottery $y_i = x_i + \epsilon_i$. Thus $y \geq x$. We restrict the domain of possible utility to be $x_i \in (-\infty, 1]$.

If the distortion preserves FOSD we should have $U^\delta(x) \leq U^\delta(y)$.

$$U^\delta(x) = \frac{\sum_{i=1}^n v(x_i)x_i p_i}{\sum_{i=1}^n v(x_i)p_i}, \tag{8}$$

$$U^\delta(y) = \frac{\sum_{i=1}^n v(y_i)y_i p_i}{\sum_{i=1}^n v(y_i)p_i}. \tag{9}$$

By first order approximation of v around x in direction $\epsilon \geq 0$.

$$U^\delta(y) = \frac{\sum_{i=1}^n v(x_i)x_i p_i + (v(x_i) + v'(x_i)x_i)p_i\epsilon_i + v'(x_i)\epsilon_i^2 p_i}{\sum_{i=1}^n v(x_i)p_i + v'(x_i)\epsilon_i p_i}. \tag{10}$$

Since $x_i \leq 1$, $\sum_{i=1}^n v(x_i)x_i p_i \leq \sum_{i=1}^n v(x_i)p_i$. This shows that

$$\forall u : v(u) + v'(u)u \geq v'(u) \Leftrightarrow \forall x \in (-\infty, 1]^n, \epsilon \text{ small }, p \in \Delta(\mathbb{R}^n) : U^\delta(x+\epsilon) \geq U^\delta(x). \tag{11}$$

The condition (11) always hold when $u = \bar{u} = 1$, which is the maximum utility the agent can get from the goods. So,

$$\forall u \leq 1 : v'(u) \leq \frac{v(u)}{(1-u)} \tag{12}$$

guarantee the FOSD after distortion. We require the sigthly stronger smooth version of it:

$$\forall u \leq 1 : v''(u)(1-u) \leq 2v'(u). \tag{13}$$

It places a restriction on how steep the v can be. Also, by second order approximation, it make sure that the distorted utility is convex in lotteries .

Easy examples of the distortion function that satisfy all conditions are

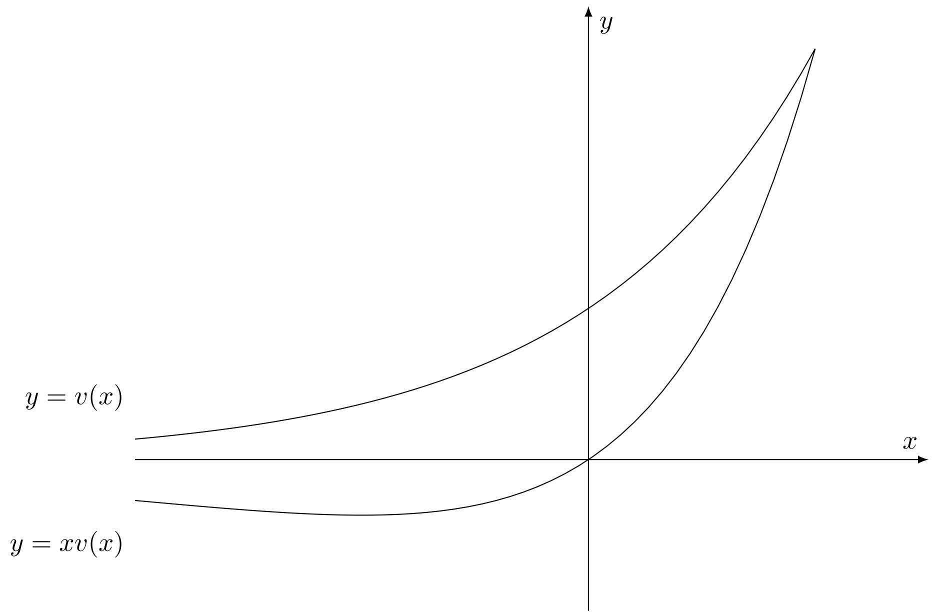
1. $v(u) = e^u + c$ when $u > 0$ and $v(u) = 1 + c$ when $u \leq 0$, for some constant c, or

2. $v(u) = au + c$, for some constant $c : c \geq a(1 - 2\underline{u})$.

3. $$v(u) = \begin{cases} e^u & \text{when } u \geq 0 \\ \frac{1}{1-u} & \text{when } u \leq 0 \end{cases}$$

With FOSD preserved distortion, the value function is increasing. Given a fixed report and mechanism, the lottery generated by the report for higher types FOSD that of lower types. Thus, given an report the utility is increasing in type, so does the upper envelope of it.

What if the condition is not satisfied, take $v(x) = e^x$ as example, the conditin fails at the negative region.



Observe that $xv(x)$ may be decreasing in some regions, it may lead to cases where the agent prefers the lotteries that are first-order stochastic dominated by the alternative lottery. The condition is just making sure that the slope and convexity of $xv(x)$ are always greater than that of $v(x)$.

A.2 Proof: The maximum spread auction gives the highest utility

Suppose the v curve is an increasing function and satisfies 3. Suppose given a distribution of utility, say $U \sim F$, and there is a mean preserve deviation of the u. We seperate U into three region U_0, U_+ and U_-. Such that for every $u_+ \in U_+, u_- \in U_-, u_+ \geq u_-$. We construct a mean preserve deviation of U by adding some constant $\Delta > 0$ to each $u_+ \in U_+$ and subtracting $\Delta \frac{F_+}{F_-}$ from each $u_- \in U_-$, where $F_+ = F(U_+)$ and $F_- = F(U_-)$. $\int_- du := \int_{u \in U_-} du$; $\int_+ du := \int_{u \in U_+} du$.

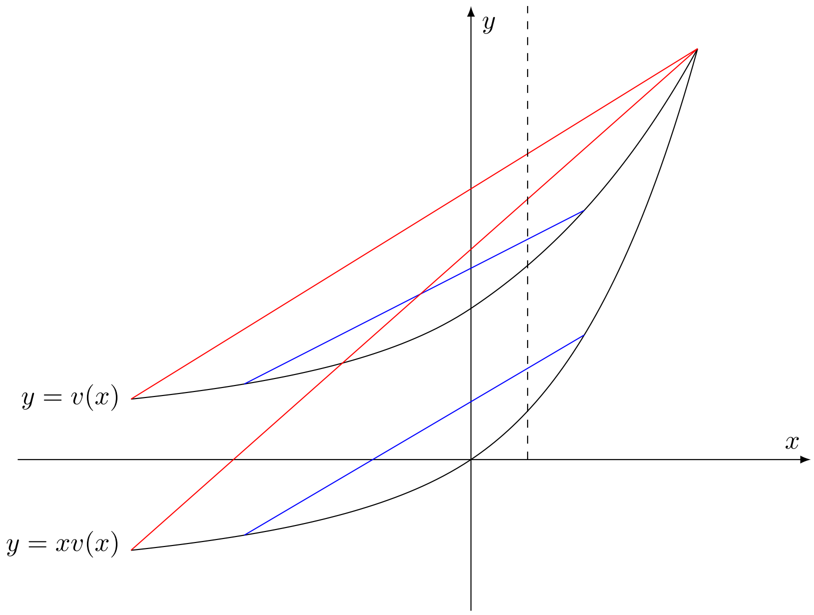
$$\begin{aligned}\frac{\int v(u')u'F(du')}{\int v(u')F(du')} &= \frac{\int v(u)uF(du) + \int (v'(u)u + v(u))(u'-u)F(du)}{\int v(u)F(du) + \int v'(u)(u'-u)F(du)} \\ &= \frac{\int v(u)uF(du) + \int_+ (v'(u)u + v(u))F(du)\Delta - \int_- (v'(u)u + v(u))F(du)\Delta\frac{F_+}{F_-}}{\int v(u)F(du) + \int_+ v'(u)F(du)\Delta - \int_- v'(u)F(du)\Delta\frac{F_+}{F_-}}\end{aligned}$$

Then by 3, $\frac{\partial^2 v(u)u}{\partial u^2} \geq \frac{\partial^2 v(u)}{\partial u^2}$, and $U_+ > U_-$. We have $\int_+ (v'(u)u + v(u))F(du)\Delta - \int_- (v'(u)u + v(u))F(du)\Delta\frac{F_+}{F_-} \geq \int_+ v'(u)F(du)\Delta - \int_- v'(u)F(du)\Delta\frac{F_+}{F_-}$, combine with the fact that $\int v(u)uF(du) \leq \int v(u)F(du)$. We know that

$$\frac{\int v(u')u'F(du')}{\int v(u')F(du')} > \frac{\int v(u)uF(du')}{\int v(u)F(du)}.$$

This means that whenever possible, by picking two regions Θ_{-i}^+ and Θ_{-i}^- with one utility greater than or equal to the other. The distorted expected utility can be increased by a mean preserved spread that increases the utility of the greater utility region Θ_{-i}^+, and reduces those of lower utility region Θ_{-i}^-.

The condition makes sure that the convexity of $xv(x)$ is always greater than v. Thus any mean preserved spread increases the distorted expected utility.



Suppose every agent has a fixed budget $t \geq 1$. This in turn implies the optimal auction must have the form:

$$Q(\theta_i, \theta_{-i}) \in \{0, 1\}$$
$$T(\theta_i, \theta_{-i}) \in \{0, t\}.$$

If there are some θ_{-i} such that $Q(\theta_i, \theta_{-i}) = 1$ and $T(\theta, \theta_{-i}) = t$ then there are no θ_{-i} such that $Q(\theta_i, \theta_{-i}) = 0$ and $T(\theta, \theta_{-i}) = 0$.

We let

$$G[\theta_{-i} : Q(\theta_i, \theta_{-i}) = 1 \text{ and } T(\theta_i, \theta_{-i}) = 0] = a(\theta)$$
$$G[\theta_{-i} : Q(\theta_i, \theta_{-i}) = 0 \text{ and } T(\theta_i, \theta_{-i}) = \bar{t}] = b(\theta)$$
$$G[\theta_{-i} : Q(\theta_i, \theta_{-i}) = 1 \text{ and } T(\theta_i, \theta_{-i}) = \bar{t}] = c(\theta)$$
$$G[\theta_{-i} : Q(\theta_i, \theta_{-i}) = 0 \text{ and } T(\theta_i, \theta_{-i}) = 0] = d(\theta).$$

The logic behind is that, given an expected allocation q, and expected payment. The consequential distortion makes a mean preserved spread always preferrible. This makes there will always be a low chance of paying the maximum payment t. While the losser region is fully occupied by payment t, the optimal auction will require some region of winning to pay t also.

A.3 Generalized Proof of

1. **If** $\hat{\theta} = \theta$ **and** $u(\hat{\theta}, \theta, Q', T') \succ^s u(\hat{\theta}, \theta, Q, T)$**, then** $f_\theta(\theta, Q', T') > f_\theta(\theta, Q, T)$**;**

2. **FOSD preserved when (4);**

3. $f_\theta(\theta)$ **increases when the type increase.**

Notations: fix an type $\theta \in \Theta$, mechanism (Q, T) and (Q', T'),

$$
\begin{aligned}
q(\theta_{-i}) :=& Q(\hat{\theta} = \theta, \theta_{-i}), & q'(\theta_{-i}) :=& Q'(\hat{\theta} = \theta, \theta_{-i}), \\
t(\theta_{-i}) :=& T(\hat{\theta} = \theta, \theta_{-i}), \text{ and} & t'(\theta_{-i}) :=& T'(\hat{\theta} = \theta, \theta_{-i}). \\
u(\theta_{-i}) :=& u(\hat{\theta} = \theta, \theta_{-i}; \theta, Q, T), & u^e(\theta_{-i}) :=& u(\hat{\theta} = \theta, \theta_{-i}; \theta + \epsilon, Q, T), \\
u'(\theta_{-i}) :=& u(\hat{\theta} = \theta, \theta_{-i}; \theta, Q', T'), \text{ and} & u'^e(\theta_{-i}) :=& u(\hat{\theta} = \theta, \theta_{-i}; \theta + \epsilon, Q', T').
\end{aligned}
$$

$\forall x_{-i} : \Theta_{-i} \to \mathbb{R}$: denote $\int_{-i} x_{-i} := \int_{\theta_{-i} \in \Theta_{-i}} x(\theta_{-i}) G(d\theta_{-i})$. Also, $\phi(u) := v(u)u$.

$$
\begin{aligned}
\mu(\theta) :=& \frac{\int_{-i} \phi(u_{-i})}{\int_{-i} v(u_{-i})}, & \mu(\theta + \epsilon) :=& \frac{\int_{-i} \phi(u^e_{-i})}{\int_{-i} v(u^e_{-i})} \\
\mu'(\theta) :=& \frac{\int_{-i} \phi(u'_{-i})}{\int_{-i} v(u'_{-i})}, \text{ and} & \mu'(\theta + \epsilon) :=& \frac{\int_{-i} \phi(u'^e_{-i})}{\int_{-i} v(u'^e_{-i})}.
\end{aligned}
$$

We study variation in mechanism (Q, T), (Q', T') mean-preserving spread of (Q, T)

for type θ, that $q' \succ^s q$, and $u' \succ^s u$.

We define the following variation funciton:

$$u^e - u := \epsilon \eta^e, \qquad u' - u := \epsilon' \eta'_{-i}, \text{ and}$$
$$u'^e - u' := \epsilon \eta'^e. \qquad \text{Those implies } u'^e - u = \epsilon' \eta' + \eta'^e \epsilon.$$

By definition $\eta'^e = q'$, and $\eta^e = q$.

We take first order approximation of ϕ and v around u, with variation η's as defined as above. For $u^x = u + \eta^x \epsilon$, for small ϵ. $\frac{\int_{-i} \phi(u^x_{-i})}{\int_{-i} v(u^x_{-i})} = \frac{\int_{-i} \phi(u_{-i}) + \int_{-i} \phi'(u_{-i}) \eta^x_{-i} \epsilon}{\int_{-i} v(u_{-i}) + \int_{-i} v'(u_{-i}) \eta^x_{-i} \epsilon}$. With for random variable X, and Y, let C be the covariance function $C(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$. Note that $\eta' = u' - u$, and u' is the mean-preserving variation of u, $\mathbb{E}\eta' = \int_{-i} \eta'_{-i} = 0$. Also, $\eta'^e = q'$ and $\eta^e = q$, with q' is an mean preserving spread of q, $\mathbb{E}\eta'^e = \mathbb{E}q' = \mathbb{E}q = \mathbb{E}\eta^e$.

We have

$$\mu(\theta) = \frac{a}{b}, \qquad \mu(\theta + \epsilon) = \frac{a + e_1}{b + e_2},$$
$$\mu'(\theta) = \frac{a + e_3}{b + e_4}, \text{ and} \qquad \mu'(\theta + \epsilon) = \frac{a + e_3 + e_5}{b + e_4 + e_6},$$

where $e_1 = [\mathbb{E}\phi'(u)\mathbb{E}q + C(\phi'(u), \eta^e)]\epsilon$, $e_2 = [\mathbb{E}v'(u)\mathbb{E}q + C(v'(u), \eta^e)]\epsilon$, $e_3 = C(\phi'(u), \eta')\epsilon'$, $e_4 = C(v'(u), \eta')\epsilon'$, $e_5 = [\mathbb{E}\phi'(u)\mathbb{E}q + C(\phi'(u), \eta'^e)]\epsilon$, and $e_6 = [\mathbb{E}v'(u)\mathbb{E}q + C(v'(u), \eta'^e)]\epsilon$.

Proof that $f_\theta(\theta|Q', T') > f_\theta(\theta|Q, T)$.

By definition $f_\theta(\theta|Q, T) = \lim_{\epsilon \to 0} \frac{\mu(\theta+\epsilon) - \mu(\theta)}{\epsilon}$, and $f_\theta(\theta|Q', T') = \lim_{\epsilon \to 0} \frac{\mu'(\theta+\epsilon) - \mu'(\theta)}{\epsilon}$.

Thus, we show that $f_\theta(\theta|Q', T') \geq f_\theta(\theta|Q, T)$, adn ignore the 2nd order ϵ, ϵ' terms.

This turn out requires

$$a(e_6 - e_2) < b(e_5 - e_1).$$

$b > a$ since $u \leq 1$. The sufficent condition for the condition becomes

$$\begin{aligned}
& e_5 - e_1 > e_6 - e_2 \\
\Leftrightarrow \quad & C(\phi'(u) - v'(u), \eta'^e) > C(\phi(u) - v'(u), \eta^e), \\
\Leftrightarrow \quad & C(\phi'(u) - v'(u), \eta'^e - \eta^e) > 0, \\
\Leftrightarrow \quad & C(\phi'(u) - v'(u), q' - q) > 0, \\
\Leftrightarrow \quad & \mathbb{E}\{(\phi'(u) - v'(u))(q' - q)\} > (\mathbb{E}\{\phi'(u) - v'(u)\})(\mathbb{E}q' - \mathbb{E}q), \\
\Leftrightarrow \quad & \mathbb{E}\{(\phi'(u) - v'(u))(q' - q)\} > 0.
\end{aligned}$$

Recall that by (3), $\phi'(u_i) - v'(u_i)$ is increasing in u_i, and (Q', T') is a mean-preserving spread of (Q, T) for θ, thus, for u_{-i} large $q' - q$ is also large. In general, the order of $\phi' - v'$ in line with $q' - q$. This shows that $C(\phi'(u) - v'(u), q' - q) > 0$. ■

Generalized A.1.: Preference is monotonic with respect to FOSD when (4)

Note that in previous we have $\eta^e_{-i} = q_{-i} \geq 0$; if we replace it by some function that is weakly positive, $\eta^e_{-i} = \eta_{-i} \geq 0$. $u^e = u + \eta\epsilon$ becomes any random utility that FOSD u by same variation in the direction η. Pick any η that is weakly positive and let

$\eta^e = \eta$. We have to proof $\lim_{\epsilon \to 0} \frac{\mu(\theta+\epsilon)-\mu(\theta)}{\epsilon} \geq 0$.

$$\begin{aligned} & \mu(\theta+\epsilon) \geq \mu(\theta), \\ \Leftrightarrow \quad & be_1 \geq ae_2. \end{aligned}$$

Since $b \geq a$, $e_1 \geq e_2$ is suffice.

$$\begin{aligned} e_1 \geq & e_2, \\ \Leftrightarrow \quad \mathbb{E}\phi'(u)\mathbb{E}\eta + C(\phi'(u), \eta) \geq & \mathbb{E}v'(u)\mathbb{E}\eta + C(v'(u), \eta), \\ \Leftrightarrow \quad \mathbb{E}\{(\phi'(u) - v'(u))\eta\} \geq & 0. \end{aligned}$$

This is direct since $\forall \theta_{-i}, \eta(\theta_{-i}) \geq 0$ and by (4), $\forall u_i : \phi'(u_i) \geq v'(u_i)$.

Generalized proof of A.2.: $\mu'(\theta) \geq \mu(\theta)$. Change of distorted utility for type θ from replacing the mechanism (Q, T) by a variation mechanism (Q', T') that is mean-preserving spread of (Q, T) for θ. By definition, we are studying small variations:

$$\begin{aligned} \lim_{\epsilon' \to 0} \frac{\mu'(\theta) - \mu(\theta)}{\epsilon'} \geq & 0 \\ \Leftrightarrow \quad be_3 \geq & ae_4 \end{aligned}$$

This is ture since $b \geq a$. $e_3 \geq e_4$ is suffice. This is automatic by (4).

A.4 Proof of Lemma 5: Given a mechanism (Q, T) that give $V(\theta), f_\theta(\theta)$, the designer can reduce $V(\theta), f_\theta(\theta)$ by reducting $Q(\theta, \cdot)$ or increasing $T(\theta, \cdot)$. Both have different relatve effect on $V(\theta)$ and $f_\theta(\theta)$.

Fix an mechanism (Q, T) and a report $\hat{\theta}$ and a type θ. Let q, t be $q(\cdot) = Q(\hat{\theta}, \cdot)$, $u(\cdot) = Q(\hat{\theta}, \cdot)\theta - tT(\hat{\theta}, \cdot)$, $v(\cdot) = v(u(\cdot))$, and $v'(\cdot) = \frac{\partial v}{\partial u}(\cdot)$. Instead of studying the effect of deviation of q, t on V, f_θ. We study the effect of (u, q) on V, q and use total derivative to find the effect of (q, t) on V, f_θ. Define the functional $V(u) = \frac{\mathbb{E}_{-i}\{uv\}}{\mathbb{E}_{-i}\{v\}}$ and $f_\theta(u, q) = \frac{\mathbb{E}_{-i}\{q(v+uv')\}\mathbb{E}_{-i}\{v\} - \mathbb{E}_{-i}\{qv'\}\mathbb{E}_{-i}\{uv\}}{(\mathbb{E}_{-i}\{v\}^2)}$; while $\hat{V}(q, t) = V(\theta q - t)$ and $\hat{f}_\theta(q, t) = f_\theta(q\theta - t, t)$.

Denote $B_1(u, q) = \mathbb{E}_{-i}\{q(v + uv')\}$, $B_2(u, q) = \mathbb{E}_{-i}\{v\}$, $B_3(u, q) = \mathbb{E}_{-i}\{qv'\}$, $B_4(u, q) = \mathbb{E}_{-i}\{uv\}$, and define the integrand $b_i(\alpha, \beta)$ by $B_i(u, q) = \int_{\theta_{-i} \in \Theta_{-i}} b_i(u(\theta_{-i}), q(\theta_{-i}))G(d\theta_{-i})$. Also let $B = B_1B_2 - B_3B_4$Using the notation we have $V = \frac{B_4}{B_2}$, $f_\theta = \frac{B}{(B_2)^2}$.

For functional F that take functions a, b as input, and deviation function η, denote the deviative of F with respect to a by function η as $\frac{\delta F}{\delta a}(a, b; \eta) = F_a^\eta(a, b) = \lim_{\epsilon \to 0} \frac{F(a+\eta\epsilon, b) - F(a, b)}{\epsilon}$.

By A.1., we have for all $\eta \geq 0$, $V_u^\eta(u) > 0$, and by A.3. We have for all $\eta \geq 0$, $f_{\theta,u}^\eta(u, q) \geq 0$. Both are ture since if $\eta \geq 0$, $u + \eta$ FOSD u. By chain rule we have

$$\begin{aligned}\hat{V}^{\eta}_{q}(q,t) =& \theta V^{\eta}_{u}(u)\\ \hat{V}^{\eta}_{t}(q,t) =& -V^{\eta}_{u}(u)\\ \hat{f}^{\eta}_{\theta,q}(q,t) =& \theta f^{\eta}_{\theta,u}(u,q) + f^{\eta}_{\theta,q}(u,q)\\ \hat{f}^{\eta}_{\theta,t}(q,t) =& -f^{\eta}_{\theta,u}(u,q)\end{aligned}$$

, where $u = q\theta - t$.

Note that B_2 and B_4 are independent of q (with u fixed):

$$f^{\eta}_{\theta,q} = \frac{B^{\eta}_{1,q}B_2 - B^{\eta}_{3,q}B_4}{(B_2)^2}.$$

Recall that $B_2(u,q) = \mathbb{E}_{-i}\{v\}$ and $B_4 = \mathbb{E}_{-i}\{vu\}$, as we have $u \leq 1$, we know that $B_2 \geq B_4$. Also, $B^{\eta}_{1,q}(u,q) = \mathbb{E}_{-i}\{\eta(v + uv')\}$ and $B^{\eta}_{3,q}(u,q) = \mathbb{E}_{-i}\{\eta v'\}$. For $\eta \geq 0$, (4) implies $B^{\eta}_{1,q} \geq B^{\eta}_{3,q}$. Thus, we have

$$\forall \eta \geq 0 : f^{\eta}_{\theta,q} \geq 0.$$

Summarizing we have for every $\eta^q \leq 0$, $\eta^t \geq 0$.

$$\begin{aligned}\hat{V}^{\eta^q}_{q}(q,t) =& \theta V^{\eta}_{u}(u) \leq 0;\\ \hat{V}^{\eta^t}_{t}(q,t) =& -V^{\eta^t}_{u}(u) \leq 0;\\ \hat{f}^{\eta^q}_{\theta,q}(q,t) =& \theta f^{\eta^q}_{\theta,u}(u,q) + f^{\eta^q}_{\theta,q}(u,q) \leq 0;\\ \hat{f}^{\eta^t}_{\theta,t}(q,t) =& -f^{\eta^t}_{\theta,u}(u,q) \leq 0.\end{aligned}$$

Also, event it two variation have the same impact on utility, the relative effect is different

$$\frac{\hat{V}_q^{\theta\eta}}{\hat{f}_{\theta,q}^{\theta\eta}} \neq \frac{\hat{V}_t^{-\eta}}{\hat{f}_{\theta,t}^{-\eta}}.$$

B Proof: The Motivating Example

Results for auctions with SEU bidders are commonly known. For FPA, both bidders bids according to strategy $\beta_i^{FPA,SEU} = \frac{1}{2}\theta_i$. For SPA, both bidders bid their own value $\beta_i^{SPA,SEU} = \theta_i$. Both revenue are equivalent.

Now, consider the case of biased bidders. The correct belief on other's type is $\theta_j \sim G = U[0,1]$, with the density $g(\theta_j) = 1$ for every $\theta_j \in [0,1] := \Theta_j$. Cosider a simple distortion that place probability δ on the best case that the other have zero valuation. Formally, the distorted belief $G^\delta : \Theta_j \to [0,1]$ is

$$G^\delta(\theta_j) = \delta \text{ for } \theta_j = 0, \text{ and } g^\delta(\theta_j) = 1 - \delta \text{ for } \theta_j \in (0,1],$$

where g^δ is the denesity of the distorted belief G^δ when $\theta_i \in (0,1]$.

Consider the symmetric Bayesian Nash equilibrium (BNE) in an FPA. To make sure equilibrium exists, we assume that the minimum bid is $\underline{b}$ and consider the limit case that $\underline{b} \to 0$.

Given θ_1 and player 2's strategy $\beta_2(\theta_{-i})$ the expected utility for bidding $b_1 \geq \underline{b}$ is

$$\begin{aligned} u_1(b_1|\theta_1) &= G^{\delta}(\beta_2(\theta_2) < b_1)(\theta_1 - b_1), \\ &= G^{\delta}(\theta_2 < \beta_2^{-1}(b_1))(\theta_1 - b_1), \\ &= (1-\delta)\beta_2^{-1}(b_1)\theta_1 - (1-\delta)\beta_2^{-1}(b_1) + \delta(\theta_1 - b_1). \end{aligned}$$

FOC gives

$$(1-\delta)\beta_2'(\beta_2^{-1}(b_1))^{-1}\theta_1 = (1-\delta)[\beta_2'(\beta_2^{-1}(b_1))^{-1}b_1 + \beta_2^{-1}(b_1)] + \delta.$$

By the symmetry $\beta_1 = \beta_2 = \beta$, and $b_1 = \beta_1(\theta_1) = \beta(\theta)$. Thus, the FOC condition becomes

$$\begin{aligned} \theta - \frac{\delta}{1-\delta} &= \beta'(\theta)\theta + \beta(\theta) = \frac{d}{d\theta}\beta(\theta)\theta, \\ \beta(\theta)\theta &= \int_0^{\theta}(x - \frac{\delta}{1-\delta})dx, \\ \beta(\theta) &= \frac{1}{2}\theta - \frac{\delta}{1-\delta}. \end{aligned}$$

Thus, the BNE symmetric equibrium is

$$\beta^{FPA,WT}(\theta) = \begin{cases} \frac{1}{2}\theta - \frac{\delta}{1-\delta} & \text{if } \theta \geq \underline{b} \text{ and } \theta \geq \frac{2\delta}{1-\delta}, \\ \underline{b} & \text{if } \theta \geq \underline{b} \text{ and } \theta < \frac{2\delta}{1-\delta}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\beta^{FPA,WT} < \beta^{FPA,SEU}$, and the difference vanishes as $\delta = 0$. Thus, the revenue of the seller reduces when the bidders are subject to WT bias.

In SPA, the strategy $\beta(\theta) = \theta$ is weakly dominant regardless of belief.