Multi-battle Group Contests

Yichuan Cai[1] Sudipta Sarangi[2]

Abstract

We consider a situation in which two groups compete in a series of battles with complete information. Each group has multiple heterogenous players. The group who first wins a predetermined number of battles wins a prize which is a public good for the winning group. A *discriminatory state-dependent contest success function* will be employed in each battle. We found that in the subgame perfect Nash equilibrium (equilibria), the lower valuation players can only exert effort in earlier battles, while the higher valuation players may exert effort throughout the entire series of battles. The typical discouragement effect in a multi-battle contest is mitigated when players compete as a group. We also provide two types of optimal contest designs which can fully resolve the free-rider problem in group contests. The intermediate prize and weighted battle scenarios are considered in the extensions.

JEL Codes: C72, D72, D74, D82, H41
Keywords: Group contest, Multi-stage, Free rider, Discouragement, Contest design, Biased contest

1 Introduction

Group contests are situations in which several groups compete for a prize through group efforts. A group effort is converted by individuals' efforts within the group through an impact function. The effort in the real world can be physical exertion in sports tournaments; campaign funds in political elections; army force in military wars; or R&D investments in patent races. The prize in a group contest is generally a public good for the winning group. In other words, every player in the winning group can enjoy the prize, regardless of her effort. In many group contests, the winner is not decided by one single battle, but by a series of battles. For instance, groups compete in a sequence of battles, and the group that first wins a certain number of battles secures the prize. We call a contest with such a structure a *multi-battle group contest*.

Examples of the multi-battle group contests are not uncommon in the real world. For instance, the NBA Finals is a best-of-seven basketball contest in which the team that first wins four games becomes the champion. In the American Primaries, the candidate and her campaign team work together to win

[1]Department of Economics, Virginia Tech, Blacksburg, VA 24061-0316. Email cyc@vt.edu
[2]Department of Economics, Virginia Tech, Blacksburg, VA 24061-0316. Email ssarangi@vt.edu

the majority of states for the nomination in their party. In WWII, two opposing military alliances – the Allies and the Axis powers – were fighting each other on battlefields globally. Apple Inc. and Samsung Electronics are in the patent race to develop the next generation, scrollable smartphones. These companies are also famous for their long-running patent battle for the smartphone and tablet design which involved multi-national litigation.

We examine a two-group multi-battle contest with complete information. All battles are carried out sequentially and the group that first wins a predetermined number of battles secure a prize which is a public good for the winning group. Groups can be heterogenous by the number of players; the required winning battles; and the players' prize valuations. In each battle, players simultaneously and independently exert their efforts. A player's effort is perfectly substitutable in the same group which means the summation of the players' efforts within a group in a component battle represents the group's effort. The winner in a component battle is decided by a discriminatory state-dependent Contest Success Function (CSF). This means the battle at each state can employ different contest frameworks and can favor a group at a state by discriminatory instruments. Examples of discriminatory state-dependent CSF are not rare in the real world. In an NBA playoff series, the home team is considered to have an advantage, meaning that it is easier for them to win the game than the away team. Similarly, in the presidential primaries, if some states prefer one specific candidate, the candidate has a higher probability of winning those states with less effort. In a patent race, the firm that leads a certain number of battles may get a head start, making it harder for the lagged firm to catch up.

We consider two main discriminatory instruments which will convert a group's effort to its effective output: 1) the additive head starts that adds to a group's effort and 2) the multiplicative bias that scales the group's effort. Two commonly adopted contest frameworks are studied: all-pay auctions (Hillman and Riley, 1989 and Baye et al., 1996) and lottery contests (Tullock, 1980). An all-pay auction is a deterministic CSF, which means the group with the larger effective output in a battle wins the battle with a probability of one. A lottery contest is a stochastic CSF, so, unlike the all-pay auctions, the group with the smaller effective output still has a chance to win the battle but lower than the other group.

The unbiased one-shot group contests with perfectly substitutable impact functions have been well studied with both deterministic and stochastic CSFs (see Baik et al., 2001; Baik, 2008; and Topolyan, 2014). The severe free-rider problem arises in their studies. More specifically, only the highest valued player(s) in each group may be active in the contest. Other players free ride on the strongest player(s) in the same group by exerting no effort. The free-rider problem also exists with other group impact functions. For example, Chowdhury et al. (2013a) studies best-shot group contests with the stochastic CSF. With the best-shot impact function, the highest effort in a group represents the group's effort. They found that a lower valued player may be active in Nash-equilibria. However, there always exists an equilibrium in which only the highest value player in each active group exerts strictly positive effort. In any equilibrium, there is only one player active in each group. In the group contest with weakest link impact functions (see Lee, 2012), the lowest effort in each group represents the group's effort. He shows that in each equilibrium, players exert the same effort within a group, however, the maximal equilibrium effort is decided by the weakest player. More literature on one-shot group contests is available with different group impact functions, cost functions, contest technologies and information structures (e.g. Kolmar and Rommeswinkel, 2013; Barbieri et al., 2014; Topolyan, 2014; Barbieri and Malueg, 2016; Chowdhury and Topolyan, 2016; Chowdhury et al., 2016).

The study of multi-battle contests can date back to Harris and Vickers (1987). The structure of sequential multi-battle contests in our study is similar to the models in Klumpp and Polborn (2006) and Konrad and Kovenock (2009) which study a sequential multi-battle contest with two players. The former study employed lottery contests while the latter employed all-pay auctions. Both studies show that a significant discouragement effect exists if one player starts to lead in the race. In Konrad and Kovenock (2009), without an intermediate prize, a player who leads two battles will fully discourage the other player and make the latter give up the remaining battles by expending no effort. Most studies on multi-battle contests focus on individual players rather than groups. Fu et al. (2015) examines the multi-battle pairwise team contests. However, unlike our group contest study, in their model, one player in each group will be assigned to compete against the other in each battle for a private prize as well as a team prize. Every player only competes once, and the team that wins the majority of the battles wins the team prize. Fu et al. (2015) shows that past battles' outcomes will not distort future battles' outcomes.

Our study differs from the existing literature by the following fact: Every battle is a group contest in which players can choose to exert effort or stay inactive in any battle. Each battle can employ different CSFs and can be biased. We first show that the severe free-rider problem also arises in biased one-shot group contests. Then we provide a complete analysis of the subgame perfect Nash equilibria (SPNE) in the multi-battle group contests. We show that in the SPNE, more players may be active during the contest. The lower valuation player can only be active in earlier battles, and the highest valuation player may be active throughout the series of battles. The free-rider problem in one-shot group contests is mitigated under the structure of multi-battle contests. The discouragement effect in multi-battle individual contests is alleviated when players compete as a group. Especially, when the number of players in a group is greater than the remaining battles that the group needs to win, the group will never be fully discouraged during the contest.

The first part of the paper assumes that the number of battles needed for a group to win and the biased states arise naturally. This phenomenon raises the question of what if a designer can control these two variables. For example, in the organizer-designed NBA play-in series, a team with a lower rank is required to win two battles while the team with a higher rank only needs to win one battle to secure a play-off spot, the game will hold in the higher ranked team's homecourt. The designer's objective is to maximize the total expected effort given players' prize distributions in each group. Optimal one-shot individual contests with the same objective as this paper have been studied in different contest technologies and discriminatory instruments. These include the all-pay auctions with multiplicative bias (see Epstein et al., 2011); the all-pay auctions with head starts (see Li and Yu, 2012); the lottery contests with head starts (see Wasser, 2013); the lottery contests with multiplicative bias (see Franke et al., 2013); and their combinations (see Franke et al., 2018; Fu and Wu, 2020). A common result in their studies is that the designer needs to balance the battle by favoring the weaker player to maximize the total effort.

In our study, we provide two types of optimal designs that has infinite and finite battels. Both can yield the maximal total expected effort equal to the higher summation of players' prize valuation in a group. In the optimal contest design with infinity battles, the designer will have the weaker group to exhaust the stronger group by favoring the weaker group in each battle. However, the weaker group needs to win infinite battles while the stronger group only needs to win finite battles. This overturns the conventional wisdom that the advantage should only be given to the weaker side. In fact, it is a one-size fits all contest design which is optimal to a wide class of objective functions. In the finite battles' design, the designer needs to balance every battle at each state completely. The number of battles needs to win for a group is exactly equal to the number of players in the group. In both optimal designs, the free-rider problem is completely resolved. We provide the complete conditions on different CSFs, which means the designer can choose any satisfied CSF in a specific battle.

In the extension part, we explore different scenarios in the multi-battle group contests to test the robustness of the main results. In the intermediate prize case, a group can be awarded by winning each single battle. Konrad and Kovenock (2009) and Fu et al. (2015) also consider the intermediate prize. In the former, the size of the intermediate prize is the same in every battle, while the later allow the size different across the battles, however, the reward should be same in a specific battle for both teams. Our study allows the reward to be virous depend on the state and can be different across two groups at the same state. In the weighted battle's case, we assume each battle has different weight of scores and the group who first obtain a predetermined scores wins the prize. For example, in the US primaries, winning a state with more populations usually obtain the higher delegates. We find that in both the intermediate prize and weighted battles scenarios, the active players at each state are highly related to the remaining battles. This is consistent with the original model. However, in both cases, a non-fully discouraged group may exert no effort at some state due to different reasons. In the intermediate prize's case, a group may lose a state on purpose to get a better position for winning higher intermediate prizes. In the weighted battle's case, a battle's weight may be too small to attract a group to expend effort to win the battle. Moreover, we found that increasing or decreasing a battle's weight by small amount may not change player's equilibrium strategies, however, a large enough adjustment of a battle's weight will increase or decrease the battle's intense competition. We further discuss how the main logic in multi-battle group contests in our study reflect in more one-shot group contests that we mentioned above.

The remaining sections of the paper is organized as follows. Section 2 develops the basic model set. Section 3 presents the general analysis of the Nash equilibria. Section 4 gives the optimal contest design. Section 5 presents the applications and extensions. In the last section, we give conclusions.

2 Model Set and Discriminatory state-dependent CSF

Consider two groups, group A and group B, which compete in a multi-battle group contest. Group $i \in A, B$ contains m_i risk-neutral players. The winning group will be awarded a prize which is a group-specific public good. In other words, the prize is like a public good for the winning group. The prize is valued by player k in group i as Z_{ik}, with $Z_{i1} \geq Z_{i2} \geq, \ldots, \geq Z_{im_i} > 0$, which is common knowledge. To win the prize, group i must first win T_i battles before group j wins T_j battles, where$T_i \in Z_+$ and $i \neq j$. Fig. 1 illustrates a multi-battle group contest with $T_A = T_B = 3$. Starting at the top-left corner, if group A wins a battle, the contest will move one step towards the right; if group B wins a battle, the contest will move one step towards the bottom. Group A or B wins the contest if they reach the right or bottom boundary, respectively.

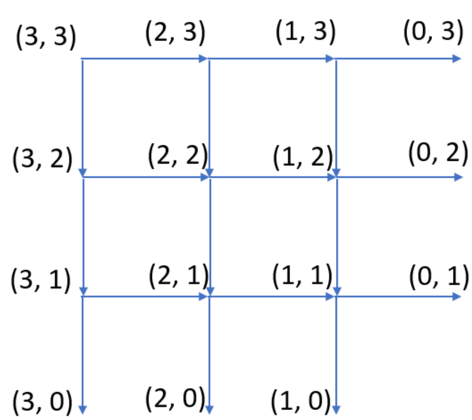

Fig. 1. A multi-battle group contest with initial state (3, 3)

At state (a, b) where $0 < a \leq T_A$ and $0 < b \leq T_B$, a component battle takes place. In each component battle, players simultaneously and independently exert effort. The expended effort is irreversible, so the effort cannot be withdrawn in the future. Let $x_{ik}(a, b) \in R_+$ represent player k's effort in group i at the state (a, b). The effort incurs a cost $c_{ik}(a, b) = x_{ik}(a, b)$. If $x_{ik}(a, b) = 0$ with a probability of one, then player k in group i is considered *inactive* at state (a, b). Otherwise, the player is *active* at state (a, b). Let $X_i(a, b)$ represent the group i's effort at the state (a, b). We assume that the player's effort is additive and perfectly substitutable in the same group, i.e., $X_i(a, b) = \sum_{k=1}^{m_i} x_{ik}(a, b)$.

Given the group's effort $X_i(a,b)$ and their rival's effort $X_j(a,b)$, the winning group at state (a,b) is decided by a discriminatory state-dependent contest success function (CSF) $p_i(a,b) = p_i\left(Y_i(a,b), Y_j(a,b)\right)$ where $Y_i(a,b) = \alpha_i(a,b) X_i(a,b) + \delta_i(a,b)$ with $\alpha_i(a,b) > 0, \delta_i(a,b) \geq 0$. The $\alpha_i(a,b)$ is a multiplicative bias and the $\delta_i(a,b)$ is an additive head starts, and they work together to convert group i's effort to an effective output at state (a,b). The commonly adopted assumption in group contest literature is when $\alpha_i(a,b) = 1$ and $\delta_i(a,b) = 0$ for both groups. We loosen the assumption and allow the effective output function to be discriminated (e.g.,$(\alpha_i(a,b), \delta_i(a,b)) \neq (\alpha_j(a,b), \delta_j(a,b))$) and state-dependent (e.g., $(\alpha_i(a,b), \delta_i(a,b)) \neq (\alpha_i(a',b'), \delta_i(a',b'))$ where $(a,b) \neq (a',b')$). The example of discriminatory state-dependent CSF is not uncommon in the real world. In an NBA playoff series, the home team is considered to have an advantage, meaning that it is easier for them to win the game than the away team. Similarly, in the presidential primaries, if some states prefer one specific candidate, the candidate has a higher probability of winning those states with less effort. In a patent race, the firm that leads a certain number of battles may get a head start, making it harder for the lagged firm to catch up.

In our study, we explore the two most famous contest frameworks: lottery contests and all-pay auctions.The lottery contests are stochastic, meaning the group with a higher effective output has a higher winning probability, but still can lose the state. The all-pay auctions are deterministic, which means the group with a higher effective output wins the state with a probability of one. The probability that group i wins state (a,b) can be expressed as follows:

1. Lottery contest with bias and head starts:

$$p_i(a,b) = \begin{cases} \frac{Y_i(a,b)}{Y_i(a,b)+Y_j(a,b)} & \text{if } Y_i(a,b) + Y_j(a,b) > 0 \\ \frac{1}{2} & \text{if } Y_i(a,b) + Y_j(a,b) = 0 \end{cases} \tag{1}$$

2. All-pay auction with bias and head starts:

$$p_i(a,b) = \begin{cases} 1 & \text{if } Y_i(a,b) > Y_j(a,b) \\ \frac{1}{2} & \text{if } Y_i(a,b) = Y_j(a,b) \\ 0 & \text{if } Y_i(a,b) < Y_j(a,b) \end{cases} \tag{2}$$

To guarantee the existence of Nash equilibrium, we stipulate that the two bias instruments cannot be employed in the all-pay auctions simultaneously, i.e., $\alpha_i(a,b) = 1$ if $\delta_i(a,b) > 0$, or $\delta_i(a,b) = 0$ if $\alpha_i(a,b) \neq 1$. Note that we allow the contest frameworks to be different across the states (state-dependent). This means the lottery contests and all-pay auctions can be employed at the same time in a multi-battle group contest. In the next section, we explore the characteristics of Nash equilibrium in the multi-battle group contests.

3 The Characteristics of Equilibrium

We begin the analysis at the state (1, 1) in which both groups have only one remaining battle. The group that wins the state can secure the prize. Therefore, if the contest starts at state (1, 1), it is a general one-shot group contest. The unbiased one-shot group contest, i.e., $Y_i(a,b) = X_i(a,b)$, has been well studied with both stochastic and deterministic contest frameworks (see Baik et al., 2001; Baik, 2008 and Topolyan, 2014). The severe free-rider problem arises in their studies. More specifically, only the highest valued player(s) in each group may be active in the contest. Other players free ride on the strongest player(s) in the same group by exerting no effort. In the following, we examine if the free-rider problem still arises in the biased one-shot group contest. We first explore the all-pay auction with multiplicative bias. Lemma 1 illustrates the equilibrium strategy. For the sake of simplicity, we omit the state sign (1, 1) from each variable in Lemma 1 through Lemma 3.

Lemma 1. (All-pay auction with multiplicative bias)

Without loss of generality, we assume that $\alpha_i Z_{i1} \geq \alpha_j Z_{j1}$.

1. *In equilibrium, group's effort satisfy the following cumulative distribution functions:* $F_i(X_i) = \frac{\alpha_i X_{i1}}{\alpha_j Z_{j1}}$ *for* $X_i \in [0, \frac{\alpha_j}{\alpha_i} Z_{j1}]$; $F_j\left(X_j\right) = \frac{Z_{i1} - \frac{\alpha_j}{\alpha_i} Z_{j1} + \frac{\alpha_j}{\alpha_i} X_j}{z_{i1}}$ *for* $X_j \in [0, Z_{j1}]$.
2. *If* $Z_{i1} > Z_{i2} \geq, \ldots, \geq Z_{im_i} > 0$, *then* $Ex^*_{i1} = EX^*_i$ *and* $x^*_{ik} = 0$ *where* $k \neq 1$; If $Z_{i1} =, \ldots, = Z_{it} > Z_{it+1} \geq, \ldots, \geq Z_{im_i} > 0$, *then* $\sum_{k=1}^{t} Ex^*_{ik} = EX^*_i$ *and* $x^*_{ik} = 0$ *where* $k > t$.

According to Lemma 1, only the highest valued player(s) in each group may be active, which matches the result in the previous study. When there are multiple players who have the same prize valuation in the group with the higher maximal prize valuation, Topolyan (2014) shows the existence of a continuum of equilibria while the group effort in the equilibrium remains consistent. The continuum of equilibria also exists with the multiplicative bias; however, a further analysis of the equilibria is beyond the scope of this study. Lemma 2 shows the equilibrium strategy when employing an all-pay auction with a head start.

Lemma 2. (All-pay auction with additive head starts)

Without loss of generality, we assume that $Z_{i1} + \delta_i \geq Z_{j1} + \delta_j$.

1. *In equilibrium, group's effort satisfy the following cumulative distribution functions:* $F_i(X_i) = \frac{\delta_i - \delta_j + X_i}{Z_{j1}}$ *for* $X_i \in \left[0, Z_{j1} + \delta_j - \delta_i\right]$ *if* $\delta_j \leq \delta_i \leq Z_{j1} - Z_{i1} + \delta_j$ *and* $X_i \in \left[0, Z_{j1} + \delta_j - \delta_i\right]$*if* $\delta_i < \delta_j$; $F_j\left(X_j\right) = \frac{Z_{j1} - Z_{i1} + X_j}{Z_{j1}}$*for* $X_j \in [\delta_i - \delta_j, Z_{j1}]$ *if* $\delta_j \leq \delta_i \leq Z_{j1} - Z_{i1} + \delta_j$ *and* $X_j \in [0, Z_{j1}]$ *if* $\delta_i < \delta_j$. $X^*_i = X^*_j = 0$ *if* $\delta_i > Z_{j1} - Z_{i1} + \delta_j$.
2. *If* $Z_{i1} > Z_{i2} \geq, \ldots, \geq Z_{im_i} > 0$, *then* $Ex^*_{i1} = EX^*_i$ *and* $x^*_{ik} = 0$ *where* $k \neq 1$; If $Z_{i1} =, \ldots, = Z_{it} > Z_{it+1} \geq, \ldots, \geq Z_{im_i} > 0$, *then* $\sum_{k=1}^{t} Ex^*_{ik} = EX^*_i$ *and* $x^*_{ik} = 0$ *where* $k > t$.

Similar to Lemma 1, the severe free-rider problem still exists with the additive head starts. Note that in the unbiased group contest, the active player in the group with lower maximal prize valuation will be fully exhausted and have 0 expected payoff. This happens to the active player in the group with the lower product of bias and prize valuation or with the lower summation of head starts and prize valuation. We will use this property to design an optimal group contest in the next section. Lemma 3 illustrates the equilibrium strategies in a lottery contest with both bias and head starts.

Lemma 3. (Lottery contest with multiplicative bias and additive head starts)

1. *The equilibrium group effort satisfies* $X^*_i = \frac{\alpha_i Z^2_{i1} \alpha_j Z_{j1}}{(\alpha_i Z_{i1} + \alpha_j Z_{j1})^2} - \frac{\delta_i}{\alpha_i}$ *if* $\delta_i \leq \sqrt{\alpha_i \delta_j z_i} - \delta_j$; otherwise, $X^*_i = X^*_j = 0$.

2. *If* $Z_{i1} > Z_{i2} \geq, \ldots, \geq Z_{im_i} > 0$*, then* $x^*_{i1} = X^*_i$ *and* $x^*_{ik} = 0$ *where* $k \neq 1$; If $Z_{i1} =, \ldots, = Z_{it} > Z_{it+1} \geq, \ldots, \geq Z_{im_i} > 0$*, then* $\sum_{k=1}^{t} x^*_{ik} = X^*_i$ *and* $x^*_{ik} = 0$ *where* $k > t$.

According to Lemma 2 and Lemma 3, when a group's head start is large enough, then both groups' equilibrium efforts equal to 0. This phenomenon also arises when the multiplicative bias equals 0, which is not allowed in our assumption. We call the CSF trivial when both groups' maximal prize valuations are positive, while their group efforts are both equal to 0. In our paper, we focus on the non-trivial CSF.

Next, we explore the states beyond the state (1, 1). The solution concept we will use is subgame perfect Nash equilibrium. We denote $v_{ik}(a, b)$ the *subgame perfect continuation value* of player k in group i at state(a, b). The continuation value represents the player's equilibrium expected payoff if the contest starts at state (a, b). The continuation values at the terminal states are $v_{Ak}(0, b) = Z_{Ak}$ and $v_{Bk}(0, b) = 0$; or $v_{Ak}(a, 0) = 0$ and $v_{Bk}(a, 0) = Z_{Bk}$, where $a > 0$ and $b > 0$.At state (a, b) where $a > 0$ and $b > 0$, a component battle takes place. The continuation value for player k in group A can be represented as $v_{Ak}(a, b) = p^*_A(a, b)(v_{Ak}(a-1, b) - v_{Ak}(a, b-1)) + v_{Ak}(a, b-1) - x^*_{Ak}(a, b)$. It is apparent that the value that attracts the player to win the state is $v_{Ak}(a-1, b) - v_{Ak}(a, b-1)$. Let $z_{Ak}(a, b) = v_{Ak}(a-1, b) - v_{Ak}(a, b-1)$ and $z_{Bk}(a, b) = v_{Bk}(a, b-1) - v_{Bk}(a-1, b)$ be the player's prize spread in each group. The battle at state (a, b) reduced to a one-shot group contest with a player's prize valuation equal to her prize spread.

Before continuing, we should mention a specific tie-breaking rule to guarantee the existence of Nash equilibrium in some situations. For instance, we assume $\alpha_i(a, b) = 1 and \delta_i(a, b) = 0$ for both groups. If $\max \mathbf{z_A}(a, b) = 0$, then all players in group A remain inactive, regardless of group B's effort. If $\max \mathbf{z_B}(a, b) > 0$, then at least one player in group B has the incentive to expend the smallest effort to win the battle with a probability of one. However, the smallest number of efforts does not exist. The following specific tie-breaking rule can avoid the non-existence of Nash equilibrium in such situations: when $Y_A(a, b) = Y_B(a, b)$, if $\max \mathbf{z_i}(a, b) > 0$ and $\max \mathbf{z_j}(a, b) = 0$, then $p_i(a, b) = 1$.

According to above the analysis, we can directly apply lemma 1, 2 and 3 at each state with players' prize valuations equaling their prize spreads. The active players and expected individual's and group's effort at every state are summarized as the following Proposition.

Proposition 1.

a) Player k is active at the state (a, b) only if $z_{ik}(a, b) = \max \mathbf{z_i}(a, b) > 0$ *where* $\mathbf{z_i}(a, b) = (z_{i1}(a, b), z_{i2}(a, b), ..., z_{im_i}(a, b))$.

b) The equilibrium expected effort for group i at state (a, b) satisfies $\sum Ex^*_{it}(a, b) = EX^*_i(a, b)$ *where player t satisfies* $z_{it}(a, b) = \max \mathbf{z_i}(a, b)$.

According to Proposition 1, we can directly identify some differences between the one-shot group contests and the multi-battle group contests in equilibrium. In a one-shot group contest, if both groups only have one player with the highest prize valuation, the unique Nash equilibrium exists. However, in a multi-battle group contest with the same prize distribution, multiple subgame perfect Nash equilibria may exist if more than one player has the highest prize spread in a group. In a one-shot group contest, when multiple equilibria exist, the expected total effort remains constant in any equilibrium. In a multi-battle group contest, the total effort varies depending on the subgame perfect Nash equilibria. Corollary 1 summarizes these results.

Corollary 1.

a) If $\max \mathbf{z_i}(a, b)$ *is a singleton for every state (a, b) where* $0 < a \leq T_A$ *and* $0 < b \leq T_B$*, then the multi-battle group contest has a unique subgame perfect Nash equilibrium.*

b) When there exist multiple subgame perfect Nash equilibria, then the total effort and the winning probability for each group may vary in different equilibria.

An example of Corollary 1(b) is given in the Appendix. We next explore the relationship between the players' prize valuations and their activations at each state. We want to know what encourages or discourages a player from expending effort at a certain state. We define $P_i(a,b)$ as group i's probability to win the final prize at state (a,b). The probability can be obtained iteratively. For example, $P_A(a,b) = p_A(a,b) \cdot P_A(a-1,b) + (1-p_A(a,b)) \cdot P_A(a,b-1)$. With a slight abuse of notation, we define $\sum x_{ik}(a,b)$ as the expected total effort for player k in group i at state (a,b). The expected total effort can be represented as $\sum x_{ik}(a,b) = p_A(a,b) \sum x_{ik}(a-1,b) + p_B(a,b) \sum x_{ik}(a,b-1)$. Now, a player's continuation values can be represented as $v_{ik}(a,b) = P_i^*(a,b) Z_{ik} - \sum x_{ik}^*(a,b)$. Rewriting the prize spread equation, we have Observation 1.

Observation 1.

$$z_{Ak}(a,b) = (P_A(a-1,b) - P_A(a,b-1))Z_{Ak} - \sum x_{Ak}(a-1,b) + \sum x_{Ak}(a,b-1) \tag{3}$$

$$z_{Bk}(a,b) = (P_B(a,b-1) - P_B(a-1,b))Z_{Bk} - \sum x_{Bk}(a,b-1) + \sum x_{Bk}(a-1,b) \tag{4}$$

It can be easily proved that $P_A(a-1,b) - P_A(a,b-1) \geq 0$ and $P_B(a,b-1) - P_B(a-1,b) \geq 0$. Therefore, a higher prize valuation increases a player's prize spread at every state. Because every player in the same group faces the same winning probability at each state, the probability difference between winning and losing does nothing on deciding the active player at a state. Unlike the conventional wisdom that the future effort discourages a player in an earlier battle, only the expected effort after winning the state reduces the player's prize spread. The expected effort after the player losing the state, on the contrary, will encourage the player to win an earlier battle.

To better illustrate the active players at each state, we denote R_t^i the set that includes top t prize valuations in group i. Therefore, we have$R_{t_1}^i \subset R_{t_2}^i$if $t_1 < t_2$.For example, the highest prize valuation in group i is the only element ofR_1^i, and the highest and the second highest prize valuation in group i are elements ofR_2^i. Proposition 2 shows how player's activity at a state related to the player's prize valuation and the remaining battles for that group.

Proposition 2.

a) *Player k in group A is active at state* $(a_t,\, b)$ *only if* $Z_{Ak} \in R_{a_t}^A$, *player k in group B is active at states* $(a,\, b_t)$ *only if* $Z_{Bk} \in R_{b_t}^B$.

b) (1) *If player l in group A has the lowest prize valuation among active players at every state* $(a,\, b)$ *where* $a \leq a_t$ *and* $b \leq T_B$, *and* $Z_{Al} = \min R_t^A$, *then at state* (a_t+1, b), *player k is active only if* $Z_{Ak} \in R_{t+1}^A$. (2) *If player l in group B has the lowest prize valuation among active players at every state* $(a,\, b)$ *where* $a \leq T_A$ *and* $b \leq b_t$, *and* $Z_{Bl} = \min R_t^l$, *then at state* (a, b_t+1), *player k is active only if* $Z_{Bk} \in R_{t+1}^B$.

Proposition 2 says that when increasing a group's remaining battles, the inactive player in that group may become active by the order from high valuation to low valuation. The player with a lower prize value will not become active before the player who has a strictly higher prize value become active. Once become active, the player may remain active at the states with more remaining battles in the group. This contradicts the result in one-shot group contest that only the player with the highest prize valuation may be active. In the multi-battle group contest, more players may be active in the contest. A possible equilibrium outcome is that the players with the lowest prize valuation remain inactive at all states. The players with the relatively low prize valuation are active in earlier battles while the players with the higher prize valuation is active everywhere during the contest. The intuition behind this is that a player with a higher prize valuation is discouraged by future effort, which may cause she stay inactive in earlier battle. In those battles, the players with a relatively low prize valuation expend their effort to free ride on stronger players in future battles. However, if a player's prize valuation is high enough, it can overcome the discourage effect on future battles.

Proposition 2 does not tell the exact active player at a certain state because it depends on the prize distribution and the CSF employed at each state. However, it narrows down the possible equilibrium outcome and excludes a lot of impossible cases. Figure 2 shows some examples of possible and impossible cases of active players in group A. The number stands for the active player in group A at each state. For the sake of simplicity, we assume that $Z_{A1} > Z_{A2} > Z_{A3}$. We can see that varied on the CSF and the prize distribution, player 1 may be active at every state or only active when there is only one remaining battle to win. For player 3, she will never be active when there are less than three remaining battles.

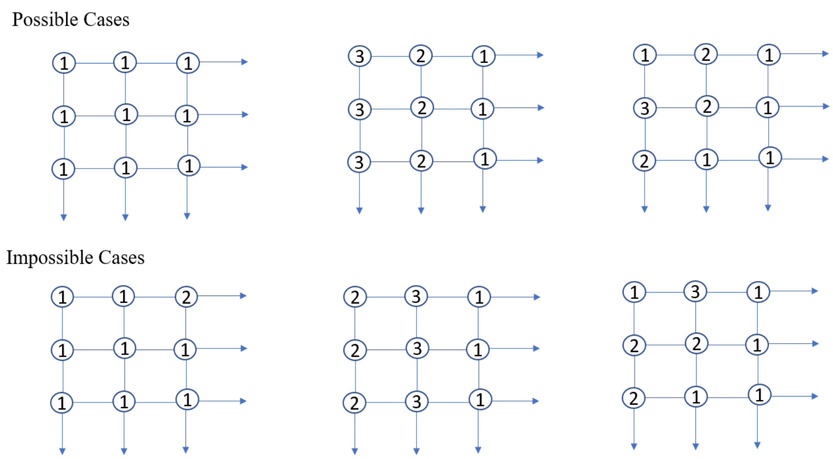

Figure 2. Possible Active Players at each State in Group *A*

In the possible cases of figure 2, players are active at every state. This raises a natural question: is that possible that there is no active player at some states? In the multi-battle contests with individual players, previous literature has shown that one player's large lead may fully discourage the other player and make the lagging player give up the contest by staying inactive in the remaining battles. For example, Konrad et al. (2009) show that without an intermediate prize, a player who is behind two battles will expend no effort in the remaining battles and will lose the contest with a probability of one. This raises another question: will a group be fully discouraged in the multi-battle group contest? To investigate this, we first need to know how each group's remaining battles affect the player's continuation values and their prize spread. Lemma 4 shows the relationships.

Lemma 4.

1. *If* $0 < v_{ik}(a,b) < Z_{ik}$*, then* $v_{Ak}(a,b-1) < v_{Ak}(a,b) < v_{Ak}(a-1,b)$*,* $v_{Bk}(a,b-1) > v_{Bk}(a,b) > v_{Bk}(a-1,b)$*.*

2. *If* $\exists v_{ik}(a,b) = Z_{ik}$*, then* $v_{it}(a,b) = Z_{it}$ $\forall t \in (1,\dots,m_i)$ *and* $v_{jk}(a,b) = 0$ $\forall k \in (1,\dots,m_j)$*.*

3. *If* $v_{ik}(a,b) = 0$*, then* $v_{Ak}(a+1,b) = v_{Ak}(a,b-1) = 0$*,* $v_{Bk}(a,b+1) = v_{Bk}(a-1,b) = 0$*.*

Lemma 4 shows that the player's continuation value will decrease by increasing their own group's remaining battles to win or decreasing the rival group's remaining battles until the continuation value becomes 0. On the contrary, the player's continuation value will increase by decreasing their own group's remaining battles to win or increasing the rival group's remaining battles until the continuation value equals the prize valuation. Note that if one player's continuation value is equal to her prize valuation at a state, then all the other players' continuation value in the same group is equal to their prize valuation. Meanwhile, all the players' continuation value in the rival group is equal to 0 at the state. We say group i is *fully discouraged* at state (a,b) if $P_i^*(a,b) = 0$. Lemma 5 illustrate the condition when a group is fully discouraged.

Lemma 5. *Group i is fully discouraged at state* (a,b) *if and only if* $\exists v_{jk}(a,b) = Z_{jk}$*.*

We say there is a *competition* at state (a,b) if both groups' effort is positive. It is not hard to see that there is no competition in the remaining battles if one group is fully discouraged in equilibrium. The occurrence of the fully discouraged group will make some states be reached without positive probability. This discouragement effect is very common in the multi-battle contests with individual players. However, the structure of multi-battle group contest can mitigate this effect.

Proposition 3.

a) If $\max \mathbf{z_A}(T_A, 1) > 0$ *and* $\max \mathbf{z_B}(1, T_B) > 0$*, then the competition arises at every state and each state is reached with positive probability if the contest starts at* (T_A, T_B)*.*

b) Group i will never be fully discouraged during the contest when $T_i \leq m_i$*.*

The formal proof is relegated to the Appendix. Proposition 3 provides two easy ways to check if a fully discouraged group may occur during the contest. According to proposition 3, the more players in a group, the more unlikely the group is fully discouraged in the contest. When the members in a group more than the initially remaining battles need to win, the group will never be fully discouraged in the contest. The existence of fully discouraged group is also based on the CSF. However, we can conclude that the minimal lead to force the lagging group to give up the contest equals the number of the lagging group's players.

4 Optimal Contest Design

In many contests, a contest designer exists whose objective is to maximize the total expected effort during the contest. A majority of literature studies the optimal design in contests with individual players (e.g., Epstein et al., 2011; Li and Yu, 2012; Wasser, 2013; Franke et al., 2013; Franke et al., 2018). The optimal head starts and the multiplicative bias has been obtained in both the all pay auctions and the lottery contests with individual players. Our study allows the contest designer to choose an initial state $T = (T_A, T_B)$ and a set of discriminatory state-dependent CSF $\mathbf{p} = \{p_i(a,b)\}$ where $i \in A, B$ and $0 < a \leq T_A$, $0 < b \leq T_B$. Therefore, the optimization problem can be written as follows:

$$\max_{\{T,\mathbf{p}\}} \sum_{i=A,B} \sum_{k=1}^{m_i} \sum x_{ik}(T_A, T_B) \;\; subject\ to\ x_{ik}(a,b) = arg \max_{x_{ik} \geq 0} v_i(a,b) \tag{5}$$

A general way to solve the optimization problem is to find each player's equilibrium effort $x_{ik}(a,b)$ at every state for an arbitrary contest design $(T, \mathbf{p})$, which is almost impossible. To avoid the complexity, we first find the upper bound of the possible total effort in equilibrium, then we investigate if there exists such contest design that can make the total effort approach to the upper bound.Because $\sum_{i=A,B}\sum_{k=1}^{m_i}\sum x_{ik}(T_A,T_B) = \sum_{i=A,B}\sum_{k=1}^{m_i}(P_i(T_A,T_B)\cdot Z_{ik} - v_{ik}(T_A,T_B))$. The following inequation shows the maximal possible total effort.

$$\sum_{i=A,B}\sum_{k=1}^{m_i}(P_i(T_A,T_B)\cdot Z_{ik} - v_{ik}(T_A,T_B)) \le \sum_{i=A,B}\sum_{k=1}^{m_i} P_i(T_A,T_B)\cdot Z_{ik} \le \max(\sum_{k=1}^{m_A} Z_{Ak}, \sum_{k=1}^{m_B} Z_{Bk}) \tag{6}$$

Without loss of generality, we assume that $\sum_{k=1}^{m_A} Z_{Ak} \sum_{k=1}^{m_B} Z_{Bk}$[3]. According to (6), $\sum_{i=A,B}\sum_{k=1}^{m_i}\sum x_{ik}(T_A,T_B) = \sum_{k=1}^{m_A} Z_{Ak}$ if and only if $P_A(T_A,T_B) = 1$ and $v_{Ak}(T_A,T_B) = 0\ \forall k \in (1,\ldots,m_A)$. These two conditions seem opposite. Typically, when a player or a group's winning probability close or equal to one, they have a huge advantage in the contest. In many cases, they only exert an extremely small or 0 effort in equilibrium. However, the second condition shows that no player in group A has a positive expected payoff. When the total effort reaches the upper bounder, these two conditions must be both satisfied. We first explore when the second condition is satisfied.

Lemma 6. *When $v_{Ak}(T_A,T_B) = 0\ \forall k \in (1,\ldots,m_A)$, then $T_A \ge m_A$. If $T_A = m_A$, then $v_{At}(t,b) = 0\ \forall t \in (1,\ldots,m_A)$ and $b \le T_B$.*

Lemma 6 can be directly obtained from lemma 4 and proposition 3. We already know that when $T_A \le m_A$, group A will never be fully discouraged during the contest. This means when $T_A < m_A$, at least one player in group A has positive expected payoff at the initial state. Therefore, the smallest available number for T_A is m_A. When $T_A = m_A$, the CSF must fully exhaust the player who has the highest prize spread in group A at each state. This means player 1 in group A's continuation value at the states where group A has only one remaining battle to win must equal 0, and player t in group A's continuation value at the states where group A has only t remaining battles to win must equal to 0. Note that when multiple players who have the same prize spread at state (t, b), without loss generality, we shall assign the player number t to the active player at state (t, b).

According to above analysis, when $T_A = m_A$, then $\max \mathbf{z_B}(a,b) > 0$ at every state in the optimal contest design. Therefore, we must have $p_B(a,b) > 0$ at every state. The question may arise: if group B has positive winning probability at every state, how can we design a contest that can make $P_A(T_A,T_B) = 1$? Lemma 7 provide a possible contest design.

Lemma 7. *When $P_A(T_A,T_B) = 1$ and $p_B(a,b) > 0\ \forall a \le T_A,\ b \le T_B$, then T_A is a finite number and $T_B \to \infty$.*

According to lemma 6 and lemma 7, we establish an optimal contest design.

Proposition 4. *When $\sum_{k=1}^{m_i} Z_{Ak} > \sum_{k=1}^{m_j} Z_{Bk}$, the contest design $(T, \mathbf{p})$ yields the maximal total effort $\sum_{k=1}^{m_i} Z_{Ak}$ if*

1. *$T_A = m_A$ and $T_B \to \infty$.*
2. *Each state employ an all-pay auction with a head start that satisfies $\delta_B(a,b) > \max \mathbf{z}_A(a,b) - \max \mathbf{z}_B(a,b) + \delta_A(a,b)$; or a multiplicative bias that satisfies $\alpha_B(a,b) > \frac{\alpha_A(a,b)\max \mathbf{z}_A(a,b)}{\max \mathbf{z}_B(a,b)}$.*

[3] We will discuss the case $\sum_{k=1}^{m_A} Z_{Ak} = \sum_{k=1}^{m_B} Z_{Bk}$ later in the section.

According to proposition 4, the optimal set of discriminatory state-dependent CSF is not unique. At each state, the designer can either employ one of the two discriminatory instruments with all-pay auctions. However, the discriminatory instruments should large enough for the active player in group B to exhaust the active player in group A. Note that in proposition 4, we do not provide the satisfied head starts or multiplicative bias when contest framework is the lottery contests. This is because employ lottery contests at any state will violates lemma 6. Proposition 4 provide an optimal contest design with the initial state that has $T_A = m_A$. In fact, this is not the unique initial state that satisfy the optimal contest design. According to lemma 6 and 7, add more states which makes $T_A > m_A$ is also available. Those added states have no CSF restriction, which means the contest designer can choose any contest frameworks and discriminatory instruments at those states.

Proposition 4 does not mention the case that when $\sum_{k=1}^{m_i} Z_{Ak} = \sum_{k=1}^{m_j} Z_{Bk}$. For example, when two groups are symmetric, say $m_A = m_B$ and $Z_{Ak} = Z_{Bk}$, then two groups have the same summation of prize valuation. In fact, in this case, we can assign any group as group A, and apply the optimal design in proposition 4 to yield the maximal effort. In the following, we provide another type of optimal contest design which has finite battles. According to inequation 1, when $\sum_{k=1}^{m_i} Z_{Ak} = \sum_{k=1}^{m_j} Z_{Bk}$, in the optimal contest design, $P_i(T_A, T_B)$ can be any value between 0 and 1 as long as all players' expected payoff at initial state is equal to 0. Proposition 4-1 provides the optimal contest design.

Proposition 4-1. When $\sum_{k=1}^{m_i} Z_{Ak} = \sum_{k=1}^{m_j} Z_{Bk}$, *the contest design* $(T, \mathbf{p})$ *yields the maximal total effort* $\sum_{k=1}^{m_i} Z_{Ak}$ *if*

1. $T_A = m_A, T_B = m_B$.
2. *Each state employ all-pay auction with head starts that satisfies* $\max \mathbf{z}_A(a,b) + \delta_A(a,b) = \max \mathbf{z}_B(a,b) + \delta_B(a,b)$*or multiplicative bias that satisfies* $\frac{\alpha_B(a,b)}{\alpha_A(a,b)} = \frac{\max \mathbf{z}_A(a,b)}{\max \mathbf{z}_B(a,b)}$.

Under the optimal contest design in proposition 4-1, the designer balanced every battle at every state. This means each group has half chance to win a battle at each state in equilibrium. Starting the initial state, every player contributes to their group. The contest design yields the same total effort as in proposition 4 but with finite battles. The contest design also delivers some interesting results even when $\sum_{k=1}^{m_i} Z_{Ak} \neq \sum_{k=1}^{m_j} Z_{Bk}$. Corollary 2 summarizes the characteristics when employing contest design in proposition 4-1.

Corollary 2. *Employing the contest design in Proposition 4-1,*

1. $P_i(T_A, T_B) = \sum_{t=0}^{m_j-1} \binom{m_i+t}{m_i} 0.5^{m_i+t}$
2. *The active player at state* (a, b) *are player a in group A and player b in group b.*

According to Corollary 2, the group with a smaller number of players has a higher winning probability of the contest but has a higher expected total effort in their group. Under the contest design, every player contributes to their group; however, if two groups' total prize valuations are not identical, then the total effort is less than the contest design in proposition 4. Another advantage of employing the optimal design is that it can easily predict the active player at each state. The rank of the active player is the same as the remaining battles that the group needs to win.

Both these two designs can fully resolve the free-rider problem. The design in proposition 4 is also optimal to other designers' objectives, such as the maximization of the winning group's effort, the stronger group's winning probability, the individual's maximal effort, the group's maximal effort, etc. The optimal design with each objective in individual contests has been well studied. Our study provides a one-size-fits-all optimal design in group contests. Note that when each group has only one player, then it is a two players' contest. Therefore, the optimal contest design can also be applied to an individual's contest. It can yield the total effort equal to the stronger player's prize valuation, which is weakly dominant to any other optimal contest design.

Another interesting research question is that if the designer can assign players to a specific group, how will the contest designer arrange the groups. According to proposition 4, the contest designer should assign all players to one group except one of the weakest players and have the weakest player exhausting all remaining players by giving the weakest player large advantages. However, the designer may not handle infinity battles (e.g., hold a battle is costly), and the designer may be required to assign the same number of players in each group (e.g., most sports games have the same number of players in each team). According to proposition 4-1, the designer needs to balance two groups by making them have the same total prize valuations.

5 Extensions

5.1 Intermediate Prize

In this section, we assume that groups not only win for the final prize, but also for the intermediate prize at every state. The counterpart of the intermediate prize in the real world can be cost reduction in the patent race, morale boost in sports competition, or prestige in the political election. The intermediate prize can be identical at every state (see Konrad et al., 2009) or be different across different battles (see Fu et al. 2015). We further loosen the assumption and allow the intermediate prize to differ at each state and between two groups. We call it a *discriminatory state-dependent intermediate prize*. For example, winning a series of matches at the homecourt can give the home team extra incentives.

Let $\beta_i(a,b)$ be the intermediate prize for group i winning the battle at state (a,b). The intermediate prize is a public good for the winning group. We assume that every player in the same group values the intermediate prize the same. Therefore, player k's continuation values in group A at state can be written as: $v_{Ak}(a,b) = p_A^*(a,b)(v_{Ak}(a-1,b) - v_{Ak}(a,b-1) + \beta_i(a,b)) + v_{Ak}(a,b-1) - x_{Ak}^*(a,b)$. With a slight abuse of notation, we define $\sum\beta_i(a,b)$ as the total expected intermediate prize for group i at state (a,b). With a slight change of observation 1, we can rewrite player k's prize spread in group A as $z_{Ak}(a,b) = (P_A(a-1,b) - P_A(a,b-1))Z_{Ak} + \sum\beta_A(a-1,b) - \sum\beta_A(a,b-1) - \sum x_{Ak}(a-1,b) + \sum x_{Ak}(a,b-1)$. Player's prize spread in group B can be written in a similar way. According to the equation of the prize spread, because the intermediate prize has the same value for players within the same group, it will not change the rank of player's prize spread at each state. Therefore, we have Corollary 3.

Corollary 3. *Proposition 1 and Proposition 2 are still hold in the case of discriminatory state-dependent intermediate prize.*

Note that the difference of the expected total intermediate prize can be negative, for example, losing a state causing a higher expected intermediate prize. This may make some or all players' prize spread negative at some states. Because the effort is a non-negative number, when all players in a group have the negative prize spread, the dominant strategy for them is to expend 0 effort. Two reasons cause a group to give up some state on purpose. First, the group is trying to choose a state where the CSF is easier for the group to win an intermediate prize. Second, the intermediate prize is larger at the state that the group loses the previous state than the group wins the previous state. In both cases, the purpose of winning an intermediate prize overcomes the benefits of winning the final prize.

5.2 Weighted Battels

Our standard model assumes that every battle has the same weight, and the group that first wins the predetermined number of battles secures the final prize. In the real world, every battle may not weigh the same. For example, in the American primaries, the candidate who wins the majority delegates will become the party's nominee. The state with more populations usually has the higher delegates. In WWII, the battle of Normandy changed the trend of the war and led the war to an end. In the Tour de France, win the different types of the mountain the race covers will get different points. In this section, we will weigh each battle differently.

We define the initial state in the contests with weighted battles as $(T_A^{'}, T_B^{'})$. $T_i^{'}$ represents the predetermined score that group i needs to obtain to secure the final prize. Let θ_t be the battle's weight or battle's score in battle t where $\theta_t \geq 0$. At the initial state, the first battle takes place. If group A wins the first battles, then the state moves to $(T_A^{'} - \theta_1, T_B^{'})$. At any state $(a^{'}, b^{'})$ where $a^{'} > 0$ and $b^{'} > 0$, a component battle takes place. We assume that at state $(a^{'}, b^{'})$, battle t takes place. If group A wins the battle, the state moves to $(a^{'} - \theta_t, b^{'})$ which mean group A only needs to earn $a^{'} - \theta_t$ scores to secure the prize. If $a^{'} - \theta_t \leq 0$, then the contest ends immediately, and group A wins the prize. Therefore, player k's continuation values in group A at state $(a^{'}, b^{'})$ when battle t takes place can be written as $v_{Ak}(a^{'}, b^{'}) = p_A^*(a^{'}, b^{'})v_{Ak}(a^{'} - \theta_t, b^{'})$ $+p_B^*(a^{'}, b^{'})v_{Ak}(a^{'}, b^{'} - \theta_t) - Ex_{Ak}^*(a^{'}, b^{'})$,where $v_{Ak}\left(a^{'} - \theta_t, b^{'}\right) = Z_{Ak}$ if $a^{'} - \theta_t \leq 0$ and $v_{Ak}\left(a^{'}, b^{'} - \theta_t\right) = 0$ if $b^{'} - \theta_t \leq 0$. The player's prize spread is $z_{Ak}\left(a^{'}, b^{'}\right) = v_{Ak}(a^{'} - \theta_t, b^{'}) - v_{Ak}(a^{'}, b^{'} - \theta_t)$. A similar analysis can be applied to group B. Therefore, proposition 1 can be directly applied to the case of weighted battles.

In the model of weighted battles, we still allow different CSFs employed in different battles. However, we require that in the same battle, the CSF should be the same. This can avoid the inconsistent of the states when we adjust a battle's weight. With the weighted battles, the number of battles needed to win is not predetermined. For example, Figure 4 illustrates a contest with initial state $(T_A^{'}, T_B^{'}) = (3, 3)$. Each battle's weight is $\theta_1 = 2$and $\theta_t = 1$ where $t = 2, 3, 4$. The red dash line represents the *impossible path*, and the red dot represents the *impossible state*. Starting from the initial state, we see that the states (2, 3), (3, 2) and (2, 2) are not reachable. A group may win the prize with two or three battles.

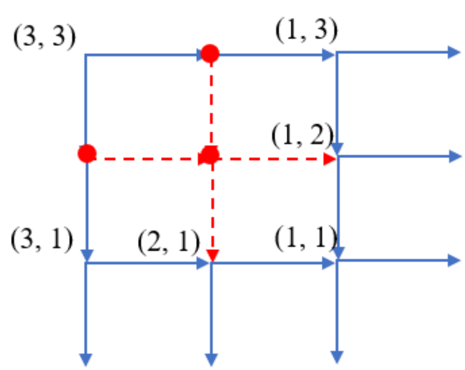

Figure 3. Weighted Battle with the Initial State $(T_A^{'}, T_B^{'})$= (3, 3)

Now we slightly change the weight of each battle and let $\theta_x = 1$where $x = 1, 2, 3$ and $\theta_4 = 2$. Figure 4 illustrates the new example. There is no impossible path in this example, and state (1, 1) is the only impossible state. Notice that at states (1, 2) and (2, 1) where battle 4 takes place, the group can secure the prize by winning the battle. Because the CSF in battle 4 is the same, the players in both groups have the same continuation values at state (1, 2) or (2, 1). We say that state (1, 2) and (2, 1) in the same *indifference continuation value segment (*ICVS*)*. More specifically, we define states $(a_1^{'}, b_1^{'})$ and $(a_2^{'}, b_2^{'})$ in the same ICVS if $a_1^{'} + b_1^{'} = a_2^{'} + b_2^{'}$ and $v_{ik}(a_1^{'}, b_1^{'}) = v_{ik}(a_2^{'}, b_2^{'})$ $\forall\, i \in (A, B)$ and $\forall k \in (1, \ldots, m_i)$. Note that at state (2, 3), winning or losing the state does not cause a difference in every player's continuation values; therefore, all players remain inactive at state (2, 3). The following Corollary tells this fact.

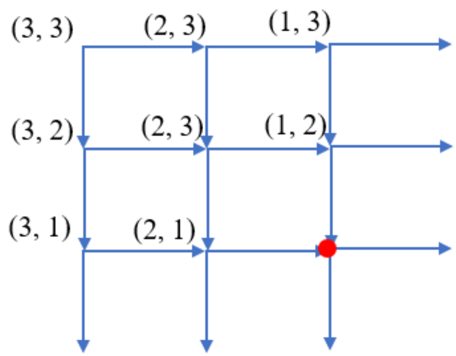

Figure 4. Weighted Battle with the ICVS

Corollary 3. *In battle t, if $(a^{'} - \theta_t, b^{'})$ and $(a^{'}, b^{'} - \theta_t)$ are in the same ICVS, then all players remain inactive at state $(a^{'}, b^{'})$.*

In the previous section, when all players remain inactive, then one of the groups is fully discouraged. Obviously, when the fully discouraged group occurs, the two states in the next battle are in the same ICVS. However, in the weighted battles' case, all players may remain inactive when both groups have the positive maximal continuation values. Lemma 5 gives the conditions that when two states are in the same ICVS.

Lemma 5. *For any two states that $a_1^{'} + b_1^{'} = a_2^{'} + b_2^{'}$,* if $\exists v_{ik}(a_1^{'}, b_1^{'}) = v_{ik}(a_2^{'}, b_2^{'}) > 0$, *then $(a_1^{'}, b_1^{'})$ and $(a_2^{'}, b_2^{'})$ are in the same ICVS.*

Lemma 5 says that when one of the players have the same positive continuation values at two states in the same battle, then all other players have the same continuation values at those two state, which means the two states are in the same ICVS.

Lemma 6. If states $(a_1^{'}, b_1^{'})$ *and $(a_2^{'}, b_2^{'})$ are in the same ICVS, and $a_2^{'} < a_3^{'} < a_1^{'}$ where $a_3^{'} + b_3^{'} = a_1^{'} + b_1^{'}$, then state $(a_3^{'}, b_3^{'})$ is in the same ICVS.*

Lemma 6 says that the ICVS are continues, which means if two states are in the same ICVS, then the states in the between of the same battle are in the same ICVS.

Lemma 7. *For any two states that $a_1^{'} + b_1^{'} = a_2^{'} + b_2^{'}$,* if $(a_1^{'}, b_1^{'})$ *and $(a_2^{'}, b_2^{'})$ are not in the same ICVS and $a_1^{'} > a_2^{'}$ then $v_{Ak}(a_1^{'}, b_1^{'}) \leq v_{Ak}(a_2^{'}, b_2^{'})$ and $v_{Bk}(a_1^{'}, b_1^{'}) \geq v_{Bk}(a_2^{'}, b_2^{'})$; $v_{ik}(a_1^{'}, b_1^{'}) = v_{ik}(a_2^{'}, b_2^{'})$ only if $v_{ik}(a_1^{'}, b_1^{'}) = 0$.*

The following graph shows an example of the ICVS. Line l_1 represents the last battle and line l_2 represents the last but one battle. The distance of each line represents the different weights of each battle. At a state in the segment AB, the last battle takes place and the group that wins the battle secures the prize. Therefore, there is no difference at a state between AB, so AB is an ICVS. We can see that CD, DE, and EF are the other three ICVS in graph 5. According to Lemma 7, for group A, the states in EF give the highest continuation values, then it is DE and CD. For group B, it is on the contrary.

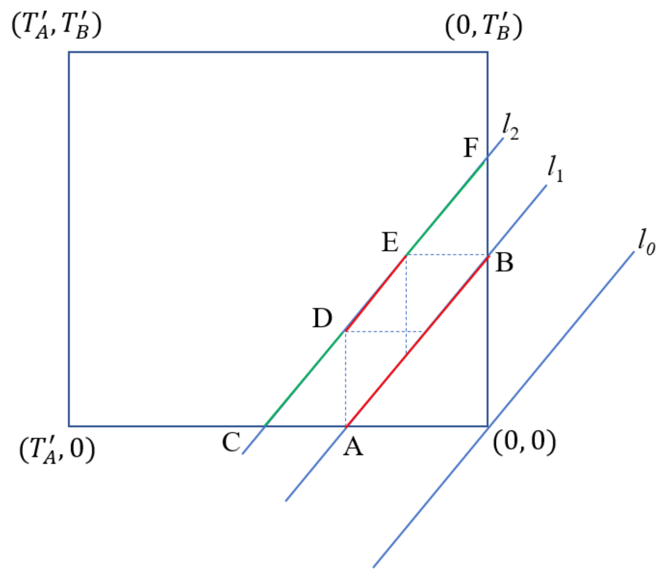

Figure 5. Weighted Battle with the ICVS

Now we investigate the effect on players' equilibrium strategies of increasing or decreasing a battle's weight. To focus on the effect of adjusting a battle's weight, we assume that other battles' weight keeps consistent. We also assume that by increasing or decreasing a battle's weight, the summation of both groups' initial score should increase or decrease by the same amount of the change of the battle's weight. This assumption can keep the continuation values consistent in future battles at the same state. For example, if battle t's weight increased with β, then the initial state should become to $(T_A^{'} + \frac{1}{2}\beta, T_B^{'} + \frac{1}{2}\beta)$. Note that when the battle's weight increases, β is positive; if the battle's weight decreases, β is negative. Other method to change the initial states, for instance $(T_A^{'} + \frac{T_A^{'}}{T_A^{'}+T_B^{'}}\delta, T_B^{'} + \frac{T_B^{'}}{T_A^{'}+T_B^{'}}\delta)$, are also satisfied. Change the initial state which satisfies the above condition will not affect the main results.

Proposition 5. *When adjust battle t's weight with β, the new initial state becomes $(T_A^{'} + \frac{1}{2}\beta, T_B^{'} + \frac{1}{2}\beta)$.*

1. *If $(a^{'} - \theta_t, b^{'})$ and $(a^{'} - \theta_t - \frac{1}{2}\beta, b^{'} + \frac{1}{2}\beta)$ in the same ICVS, and $(a^{'}, b^{'} - \theta_t)$ and $(a^{'} + \frac{1}{2}\beta, b^{'} - \theta_t - \frac{1}{2}\beta)$ in the same ICVS, then players' equilibrium strategies at state $(a^{'}, b^{'})$ keep consistent after increasing the battle's weight.*

2. *Otherwise, both groups' maximal prize spread increases by increasing the battle's weight and decreases by decreasing the battle's weight.*

Note that increasing a battle's weight will increase each player's prize spread until equal to the player's prize valuation. The battle at this state will become the same as a one-shot group contest in which the group that wins state can secure the prize. Therefore, the fully discouraged group will eventually become active when the increased weight is large enough. Decreasing a battle's weight will eventually make every player's prize spread at all states in the battle equal to 0. The main result in the weighted battles is also true in a wide class of multi-battle group contests, for example, different group impact functions, cost functions, contest technologies and information structures. The analysis of active players in the weighted battles is complicated. However, we can still conclude that more than one player may be active during the contest. The remaining battles for a group need to win at a state play an important role to decide the active player in a group, but further study is required.

5.3 Other Group Impact Functions

In the previous sections, we assume that the individual effort in the same group is additive and perfectly substitutable. Besides the perfectly substitutable impact function (IF), other group IF have also been studied in the one-shot group contests. In a weakest-link IF (Lee, 2012; Chowdhury et al., 2013a), the group's effort is equal to the lowest individual effort. In a best shot IF (Chowdhury et al., 2013a; Barbieri, 2014), the highest individual effort in a group represents the group's effort. A CES-form IF (Martin and Hendrik, 2013) is in between of the weakest-link and the best-shot IF. The player's degree of the complementarity in effort depends on the elasticity of substitution of the CES. The possibility that each group has different IF have also been studied (Chowdhury and Topolyan, 2015, 2016).

The equilibrium outcomes are various depend on the group IF in the one-shot group contest. However, the multi-battle contest structure *per se* may affect player equilibrium strategies. As the observation 1 shows, players' equilibrium strategies in a component battle in a multi-battle group contest differs from in a one-shot group contest by the following two ways. First is the difference of the whole contest winning probability that caused by winning or losing a state. In one-shot group contest, the difference is equal one. However, the difference is typically smaller in winning a state in the multi-battle group contests. This scaling down the player prize valuation and reduce their incentive to expend high effort at a lot of states. However, the scaling effect does not change the player's rank of their prize spread at each state. The second is the future efforts. We have shown that the expected effort after a player winning a state will reduce the player's prize spread while the expected effort after the player losing a state will increase the player's prize spread. Therefore, the highest valued player does not always have the higher prize spread at each state.

For example, when we employ the weakest link IF, multiple Nash equilibria exists. In each equilibrium, all players exert same effort in a group, and the maximal equilibrium effort depends on the weakest player's prize valuation. Therefore, the future effort will not distort the distribution of the prize valuation in a group. In this case, only the first effect works. The maximal equilibrium effort is reduced in each battle at most states. However, the maximal effort always depends on the player who has the lowest prize valuation. For the IF that player's equilibrium strategies are based on the player's rank or the distribution of the prize valuation, the story is different. In the one-shot IF, more than one player may active during the contest. In the CES-form IF, the player with the highest prize valuation may exert lower effort at some states. In fact, the scale effect and distort effect in a group contest with multi-battle structure can not only leads different results from a one-shot group contest by using different group IF, but also allow more variations in contest technology, cost functions, discriminatory instruments and information structures etc. The detailed effect on a specific case needs further investigation.

6 Conclusions

This paper examines a multi-battle group contest with discriminatory state-dependent CSF. Each state can employ either a stochastic or a deterministic CSF. An additive head start or a multiplicative bias can be applied to each state. We provide a complete analysis of the subgame perfect Nash equilibrium of the multi-battle group contests. We find that the lower valuation players only exert effort in earlier battles, while the higher valuation players exert effort in more pervasiveness battles. The common discouragement effect in the multi-battle contests of individuals is mitigated in the group contests. More specifically, the more players in a group, the less likely the group is fully discouraged during the contest. In the optimal multi-battle group contest design, we provide two solutions to resolve the free-rider problems completely. With the structure of infinity battles, the designer can have the weaker group fully exhaust the stronger group by favoring the weaker group in each battle. When two groups have the same summation of prize valuation, another optimal design with finite battles is available. In the optimal contest, the designer balances every battle in each state.

In the extension, we examine the intermediate prize, weighted battles, and other group IF to see if the main results are robust in different cases. In both the intermediate prize and weighted battles scenarios, each state's active players highly depend on the remaining battles. This is consistent with the original model. However, in both cases, a non-fully discouraged group may exert no effort at some state due to different reasons. In the intermediate prize's case, a group may lose a state on purpose to get a better position for winning higher intermediate prizes. In the weighted battle's case, a battle's weight may be too small to attract a group to expend effort to win the battle. Moreover, we found that increasing or decreasing a battle's weight by small amount may not change player's equilibrium strategies, however, a large enough adjustment of a battle's weight will increase or decrease the battle's intense competition. The main logic of a multi-battle group contest still holds when we employ other group IF or apply different contest technology, cost functions, and information structures. The smaller winning share and future effort together scaling and distorting the prize valuation distribution and finally affect players' equilibrium strategies at each state. The detailed effects of those variations deserve future studies.

References

[1] Kyung Hwan Baik. "Contests with group-specific public-good prizes". In: *Social choice and Welfare* 30.1 (2008), pp. 103–117.

[2] Kyung Hwan Baik, In-Gyu Kim, and Sunghyun Na. "Bidding for a group-specific public-good prize". In: *Journal of Public Economics* 82.3 (2001), pp. 415–429.

[3] Stefano Barbieri and David A Malueg. "Private-information group contests: Best-shot competition". In: *Games and Economic Behavior* 98 (2016), pp. 219–234.

[4] Stefano Barbieri, David A Malueg, and Iryna Topolyan. "The best-shot all-pay (group) auction with complete information". In: *Economic Theory* 57.3 (2014), pp. 603–640.

[5] Michael R Baye, Dan Kovenock, and Casper G De Vries. "The all-pay auction with complete information". In: *Economic Theory* 8.2 (1996), pp. 291–305.

[6] Subhasish M Chowdhury, Dongryul Lee, and Roman M Sheremeta. "Top guns may not fire: Best-shot group contests with group-specific public good prizes". In: *Journal of Economic Behavior & Organization* 92 (2013), pp. 94–103.

[7] Subhasish M Chowdhury, Dongryul Lee, and Iryna Topolyan. "The max-min group contest: Weakest-link (group) all-pay auction". In: *Southern Economic Journal* 83.1 (2016), pp. 105–125.

[8] Subhasish M Chowdhury and Iryna Topolyan. "Best-shot versus weakest-link in political lobbying: an application of group all-pay auction". In: *Social Choice and Welfare* 47.4 (2016), pp. 959–971.

[9] Subhasish M Chowdhury and Iryna Topolyan. "The Attack-And-Defense Group Contests: Best Shot Versus Weakest Link". In: *Economic Inquiry* 54.1 (2016), pp. 548–557.

[10] Gil S Epstein, Yosef Mealem, and Shmuel Nitzan. “Political culture and discrimination in contests”. In: *Journal of Public Economics* 95.1-2 (2011), pp. 88–93.

[11] Jörg Franke, Wolfgang Leininger, and Cédric Wasser. “Optimal favoritism in all-pay auctions and lottery contests”. In: *European Economic Review* 104 (2018), pp. 22–37.

[12] Jörg Franke et al. “Effort maximization in asymmetric contest games with heterogeneous contestants”. In: *Economic Theory* 52.2 (2013), pp. 589–630.

[13] Qiang Fu, Jingfeng Lu, and Yue Pan. “Team contests with multiple pairwise battles”. In: *American Economic Review* 105.7 (2015), pp. 2120–40.

[14] Qiang Fu and Zenan Wu. “On the optimal design of biased contests”. In: *Theoretical Economics* 15.4 (2020), pp. 1435–1470.

[15] Christopher Harris and John Vickers. “Racing with uncertainty”. In: *The Review of Economic Studies* 54.1 (1987), pp. 1–21.

[16] Arye L Hillman and John G Riley. “Politically contestable rents and transfers”. In: *Economics & Politics* 1.1 (1989), pp. 17–39.

[17] Tilman Klumpp and Mattias K Polborn. “Primaries and the New Hampshire effect”. In: *Journal of Public Economics* 90.6-7 (2006), pp. 1073–1114.

[18] Martin Kolmar and Hendrik Rommeswinkel. “Contests with group-specific public goods and complementarities in efforts”. In: *Journal of Economic Behavior & Organization* 89 (2013), pp. 9–22.

[19] Kai A Konrad and Dan Kovenock. “Multi-battle contests”. In: *Games and Economic Behavior* 66.1 (2009), pp. 256–274.

[20] Dongryul Lee. “Weakest-link contests with group-specific public good prizes”. In: *European Journal of Political Economy* 28.2 (2012), pp. 238–248.

[21] Sanxi Li and Jun Yu. “Contests with endogenous discrimination”. In: *Economics Letters* 117.3 (2012), pp. 834–836.

[22] Iryna Topolyan. “Rent-seeking for a public good with additive contributions”. In: *Social Choice and Welfare* 42.2 (2014), pp. 465–476.

[23] Gordon Tullock. “Efficient rent seeking”. In: *Efficient rent-seeking*. Springer, 2001, pp. 3–16.

[24] Cédric Wasser. “Incomplete information in rent-seeking contests”. In: *Economic Theory* 53.1 (2013), pp. 239–268.

APPENDIX

Proof of Lemma 1:

We assume that there is only player 1 in each group are active. Because $\alpha_i Z_{i1} \geq \alpha_j Z_{j1}$, player 1 in group i can guarantee the payoff $\alpha_i Z_{i1} - \alpha_j Z_{j1}$ when $x_{i1} = \frac{\alpha_j}{\alpha_i} Z_{j1}$ while player 1 in group j can guarantee non-negative by exert 0 effort. According to Hillman and Riley (1989), we assume that player 1 in group i exert effort according to CDF $F_i(x_{i1})$ and player 1 in group j exert effort according to CDF $F_j\left(x_{j1}\right)$, then we have $F_i\left(x_{j1}\right) Z_{j1} - x_{j1} = 0 and F_j(x_{i1}) Z_{i1} - x_{i1} = \alpha_i Z_{i1} - \alpha_j Z_{j1}$. Substitute x_{i1} and x_{j1} by $x_{i1} = \frac{\alpha_j}{\alpha_i} x_{j1}$, we have $F_i(x_{i1}) = \frac{\alpha_i x_{i1}}{\alpha_j Z_{j1}}$ for $x_{i1} \in [0, \frac{\alpha_j}{\alpha_i} Z_{j1}]$ and $F_j\left(x_{j1}\right) = \frac{Z_{i1} - \frac{\alpha_j}{\alpha_i} Z_{j1} + \frac{\alpha_j}{\alpha_i} x_{j1}}{z_{i1}}$ for $x_{j1} \in [0, Z_{j1}]$. Now we test when these two players exerting effort according to the aforementioned CDF, will other players stay inactive. The best-response for player k in group j is to maximize $\pi_{jk} = P_j\left(x_{j1} + x_{jk} > x_{i1}\right) Z_{jk} - x_{jk}$. The winning probability is as the follows:

$$P_j\left(x_{j1} + x_{jk} > x_{i1}\right) = \int_0^{Z_{j1}} f_{j1}\left(x_{j1}\right) F_i\left(x_{j1} + x_{jk}\right) dx_{j1} \tag{7}$$

$$= \frac{\left(Z_{i1} - \frac{\alpha_j}{\alpha_i} Z_{j1}\right) x_{jk}}{Z_{i1} Z_{j1}} + \int_0^{Z_{j1} - x_{jk}} \frac{\alpha_j\left(x_{j1} + x_{jk}\right)}{\alpha_i Z_{i1} Z_{j1}} dx_{j1} + \int_{Z_{j1} - x_{jk}}^{Z_{j1}} \frac{\alpha_j x_{j1}}{\alpha_i Z_{i1}} dx_{j1} \tag{8}$$

$$= \frac{\frac{1}{2}\alpha_j Z_{j1}^2 - \frac{1}{2}\alpha_j x_{jk}^2 + \alpha_i Z_{i1} x_{jk}}{\alpha_i Z_{i1} Z_{j1}} \tag{9}$$

Therefore, the first-order condition to maximize player k's payoff is $\frac{d\pi_{jk}}{dx_{jk}} = \frac{\alpha_i Z_{i1} Z_{jk} - \alpha_i Z_{i1} Z_{j1} - \alpha_j Z_{jk} x_{jk}}{\alpha_i Z_{i1} Z_{j1}} = 0$. Since π_{jk} is strictly concave in x_{jk}, $x_{jk} \geq 0$ and $Z_{j1} \geq Z_{jk}$, the best response for player k in group j is staying inactive when player 1 in group j exert effort follows $F_j\left(x_{j1}\right)$. Similarly, we have the best response for player k in group i is staying inactive when player 1 in group j exert effort follows $F_i(x_{i1})$. Therefore, player 1 in both groups play the mixed strategy $F_i(x_{i1})$ and $F_j\left(x_{j1}\right)$, and all other players use the pure strategy of 0 is a mixed strategy Nash equilibrium. Note that when there are multiple players in a group has the highest prize valuations, then each one of them can instead player 1 play the above mixed strategy. The proof of the existence of a continuum of equilibria that more than one player among the those highest valued players are active can be found in Topolyan (2014).

Proof of Lemma 2:

We assume that there is only player 1 in each group are active. Because $Z_{i1} + \delta_i \geq Z_{j1} + \delta_j$, player 1 in group i can guarantee the payoff $Z_{i1} + \delta_i - Z_{j1} - \delta_j$ by exerting effort $x_{i1} = Z_{j1} + \delta_j - \delta_i$ when $\delta_i \leq Z_{j1} - Z_{i1} + \delta_j$, and she can guarantee the payoff Z_{i1} by exerting no effort if $\delta_i > Z_{j1} - Z_{i1} + \delta_j$. Player 1 in group j can guarantee non-negative by exert 0 effort. According to Hillman and Riley (1989), we assume that player 1 in group i exert effort according to CDF $F_i(x_{i1})$ and player 1 in group j exert effort according to CDF $F_j\left(x_{j1}\right)$, then we have $F_i\left(x_{j1}\right) Z_{j1} - x_{j1} = 0$ and $F_j(x_{i1}) Z_{i1} - x_{i1} = Z_{i1} + \delta_i - Z_{j1} - \delta_j$. Substitute x_{i1} and x_{j1} by $x_{i1} = x_{j1} + \delta_j - \delta_i$, we have $F_i(x_{i1}) = \frac{\delta_i - \delta_j + x_{i1}}{Z_{j1}}$ for $x_{i1} \in \left[0, Z_{j1} + \delta_j - \delta_i\right]$ $if\ \delta_j \leq \delta_i \leq Z_{j1} - Z_{i1} + \delta_j\ and$ $x_{1i} \in \left[\delta_j - \delta_i, Z_{j1} + \delta_j - \delta_i\right] if\ \delta_i < \delta_j$; $F_j\left(x_{j1}\right) = \frac{Z_{i1} - Z_{j1} + x_{j1}}{Z_{j1}}\ for\ x_{j1} \in [\delta_i - \delta_j, Z_{j1}]\ if\ \delta_j \leq \delta_i \leq Z_{j1} - Z_{i1} + \delta_j$ $and\ x_{j1} \in [0, Z_{j1}]\ if\ \delta_i < \delta_j$.The proof of the best response of all other players is 0 is similar to lemma 1 so we omit it here.

Proof of Lemma 3:

We assume that there is only player 1 in each group are active. We have $\pi_{i1} = \frac{y_{i1}}{y_{i1} + y_{j1}} Z_{i1} - x_{i1} where y_{i1} = \alpha_i x_{i1} + \delta_i and i \in A, B$. The first order condition for maximizing π_{i1} is $\frac{\partial \pi_{i1}}{\partial x_{i1}} = \frac{y_{j1}}{(y_{i1} + y_{j1})^2} Z_{i1} - 1 = 0$. We have $y_{i1} = \sqrt{\alpha_j Z_{j1} Z_{i1}} - y_{j1}$ when $\sqrt{\alpha_j Z_{j1} Z_{i1}} - y_{j1} > 0$ and $x_{i1}^* = 0$ when $\sqrt{\alpha_j Z_{j1} Z_{i1}} - y_{j1} \leq 0$. When $\sqrt{\alpha_j Z_{j1} Z_{i1}} - y_{j1} > 0$, we have $\delta_i < \sqrt{\alpha_i \delta_j z_i} - \delta_j$. Substitute $y_j = \frac{\alpha_j Z_{j1}}{\alpha_i Z_{i1}} y_i$ to the first order condition, we have $y_{i1}^* = \frac{(\alpha_i Z_{i1})^2 \alpha_j Z_{j1}}{(\alpha_i Z_{i1} + \alpha_j Z_{j1})^2}$ and $x_{i1}^* = \frac{\alpha_i Z_{i1}^2 \alpha_j Z_{j1}}{(\alpha_i Z_{i1} + \alpha_j Z_{j1})^2} - \frac{\delta_i}{\alpha_i}$. Given player 1's strategy in each group, we examine if other player has incentive to exert positive effort. To maximize player k's payoff in group i, we have $\frac{\partial \pi_{i1}}{\partial x_{ik}} = \frac{y_{j1}}{(y_{i1} + y_{j1} + \alpha_i x_{ik})^2} Z_{ik} - 1$. Because $\frac{y_{j1}}{(y_{i1} + y_{j1})^2} = \frac{1}{Z_{i1}}$. We have $\frac{y_{j1}}{(y_{i1} + y_{j1} + \alpha_i x_{ik})^2} Z_{ik} \leq \frac{Z_{ik}}{Z_{i1}} \leq 1$. Therefore, the best response for other players other than player 1 in each group is exerting 0 effort. The proof of when there exist multiple players have the highest prize valuation can be found in Baik (2008).

An example of Corollary 1:

We assume a multi-battle contest with an initial state (2, 1). Group A has two players and group B has only one player. All of players' prize valuation is identical and equal to 1. We also assume that all state employ a general all-pay CSF. At state (1, 1), if both player in group A exert same amount of expected effort in equilibrium, then the continuation value for both players in group A at state (1, 1) is 0.25, and the player in group B has 0 continuation value at state (1, 1). Both groups have a 0.5 chance to win the state. Now we consider state (2, 1). The prize spread for players in group A are both equal to 0.25, and the player in group B has a prize spread of 1. Then group B has 0.875 chance to win the initial state.

Now we consider if there is one player in group A active at state (1, 1). Then the other player in group A has a 0.5 prize spread at state (2, 1). The player in group B's prize spread is still equal to 1. Therefore, group A's winning probability increased from 0.125 to 0.25.

Proof of Proposition 2:

We first consider the states $(1, b)$, where $b \leq T_B$. The remaining battles to win for group A is always equal to 1. If there is only one highest-valued player in group A, then player 1 is the only active player in group A at state (1, 1). Then we have $w_{A1}(1, 2) = (1 - P_A(1, 1))Z_{A1} + Ex_{A1}(1, 1) > w_{Ak}(1, 2) = (1 - P_A(1, 1))Z_{Ak}$ where $k > 1$. Let $\sum x_{ik}(a, b)$ be the total expected effort of the multi battle contest at state (a, b). Continuing the process, then we have $w_{A1}(1, b) = (1 - P_A(1, b-1))Z_{A1} + \sum x_{A1}(a, b-1) > w_{Ak}(1, b) = (1 - P_A(1, b-1))Z_{Ak}$. Therefore, if $Z_{A1} > Z_{A2}$, then player 1 is the only active player in group A at states $(1, b)$. If there are more than one highest-valued player in group A, then the player expends the highest expected effort at (1, 1) becomes the only active player at states $(1, b)$, where$b > 1$. If there are more than one player expend the highest expected effort at (1, 1), then the player who exert highest effort at (1, 2) becomes the only active player at states $(1, b)$, where$b > 2$. The players who expend lower expected effort will turn to inactive at state (1, 2). The active player will continue be eliminated until there is only one active player remained or until $b = m_B$.

Now we consider the states $(2, b)$. For the sake of simplicity, we assume that only player 1 expend positive effort at state (1, 1) and player 2 is the only second highest-valued player in group A. Then we have $w_{A1}(2, 1) = P_A(1, 1)Z_{A1} - Ex_{Ak}(1, 1)$ and $w_{A2}(2, 1) = P_A(1, 1)Z_{A2} > w_{Ak}(2, 1) = P_A(1, 1)Z_{Ak}$ where $k > 2$. We know that $\max \mathbf{w}_A(2, 1) \in (w_{A1}(2, 1), w_{A2}(2, 1))$. Therefore, player 3 through player m_A remain inactive at state (2, 1). Continuing the process, then we have $w_{A1}(2, b) = (P_A(1, b) - P_A(2, b))Z_{A1} - \sum x_{A1}(1, b) + \sum x_{A1}(2, b-1)$ and $w_{A2}(2, b) = (P_A(1, b) - P_A(2, b-1))Z_{A2} + \sum x_{A2}(2, b-1) > w_{Ak}(2, b) = (P_A(1, b) - P_A(2, b-1))Z_{Ak}$ where $k > 2$. Therefore, player 1 and player 2 may active at states $(2, b)$. Player 3 through player m_A remain inactive at states $(2, b)$.

More generally, we can represent $w_{Ak}(a, b)$ as $(P_A(a-1, b) - P_A(a, b-1))Z_{Ak} - \sum x_{Ak}(a-1, b) + \sum x_{Ak}(a, b-1)$. $(P_A(0, b) = 1 and P_A(a, 0) = 0)$ Assume that player t is active one of the states (a_t, b) where $b \leq m_b$ but is inactive at states (a, b) where $a < a_t$ and $b \leq m_b$. Then there must exist a state (a_t, b) that satisfies that $\max \mathbf{w_A}(a_t, b) = w_{At}(a_t, b) = (P_A(a_t - 1, b) - P_A(a_t, b-1))Z_{At}$. We have player k with $Z_{Ak} < Z_{At}$ is inactive at all states (a, b) where $a < a_t$. If not, then there must exists a state (a_k, b) where $a_k < a_t$ that satisfies that $\max \mathbf{w_A}(a_k, b) = w_{Ak}(a_k, b) = (P_A(a_k - 1, b) - P_A(a_k, b-1))Z_{Ak}$. However, we know that $w_{At}(a_k, b) = (P_A(a_k - 1, b) - P_A(a_k, b-1))Z_{At} > w_{Ak}(a_k, b)$. Therefore, player k is inactive at state (a_k, b). The player k is also inactive at any states of (a_t, b) because $w_{At}(a_t, b) = (P_A(a_t - 1, b) - P_A(a_t, b-1))Z_{At} + \sum x_{Ak}(a_t, b-1) > w_{Ak}(a_t, b)$.

Proof of Lemma 4:

1. Because $v_{Ak}(a, b) = p(a, b)v_{Ak}(a-1, b) - (1 - p(a, b))v_{Ak}(a, b-1)$, we have $v_{Ak}(a-1, b)$ greater

than or equal to or smaller than $v_{Ak}(a, b-1)$. We assume that $v_{Ak}(a, b-1) > v_{Ak}(a-1, b)$, then we have $v_{Ak}(a, b-2) > v_{Ak}(a-1, b-1)$ or $v_{Ak}(a-1, b-1) > v_{Ak}(a-2, b)$. Continue the process, we will have $v_{Ak}\left(a^{'}, 0\right) > v_{Ak}\left(a^{'}-1, 1\right)$ or $v_{Ak}\left(1, b^{'}\right) > v_{Ak}\left(0, b^{'}+1\right)$ where $1 < a^{'} < a$ and $1 < b^{'} < b$. However, $v_{Ak}\left(a^{'}, 0\right) = 0$ and $v_{Ak}\left(0, b^{'}-1\right) = Z_{Ak}$ which contradicting the hypothesis. Therefore, $v_{Ak}(a, b-1) \leq v_{Ak}(a-1, b)$. Now we consider the case that the continuation values are identical. When $v_{Ak}(a, b-1) = v_{Ak}(a-1, b)$ and $v_{Ak}(a, b) \neq Z_{Ak}$, then we have $v_{Ak}(a, b-2) = v_{Ak}(a-1, b-1) = v_{Ak}(a, b-2)$. Continue the process, we have $v_{Ak}\left(0, b^{'}\right) < Z_{Ak}$ and $v_{Ak}\left(a^{'}, 0\right) > 0$, which contradicting the hypothesis. Therefore, we have $v_{Ak}(a, b-1) < v_{Ak}(a, b) < v_{Ak}(a-1, b)$. The proof for $v_{Bk}(a, b)$ is similar so we omit it.

2. Because $v_{ik}(a, b) = P_i^*(a, b) Z_{ik} - \sum x_{ik}^*(a, b)$. If $v_{ik}(a, b) = Z_{ik}$, then we have $P_i^*(a, b) = 1$ and $\sum x_{ik}^*(a, b) = 0$. If $P_i^*(a, b) = 1$, then $P_j^*(a, b) = 0$ and $v_{jk}(a, b) = 0\ \forall k \in (1, \ldots, m_j)$.

Proof of proposition 5

1. If $\max \mathbf{z_A}(T_A, 1) > 0$, then $v_{Bk}(T_A, 1) < Z_{Bk}$. According to Lemma 4, $v_{Bk}(a, b) < Z_{Bk}$ where $a \leq T_A$ and $b \leq T_B$. Similarly, we have $v_{Ak}(a, b) < Z_{Ak}$ where $a \leq T_A$ and $b \leq T_B$. Because $v_{ik}(a, b) \neq Z_{ik}$, no fully discouraged group occurs during the contest, therefore, the competition arises at every state.

2. According to proposition 2, player k in group A will not turn to active at state (a, b) where $a < k$. Therefore, $v_{Ak}(a, b) > 0$ where $a < k$. Similarly, $v_{Bk}(a, b) > 0$ where $b < k$. Therefore, Group i will never be fully discouraged during the contest when $T_i \leq m_i$.